

A unified dispersion and stability analysis of discontinuous Galerkin formulations for one-dimensional elastodynamics

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Abstract

Discontinuous Galerkin (DG) discretizations of elastodynamics differ in two choices rarely compared simultaneously: which fields are interpolated, and which numerical traces couple the elements. We place Continuous Galerkin (CG) and four DG formulations of one-dimensional elastodynamics, the one-field displacement-trace (u^*) and velocity-trace (v^*) forms and the two-field displacement-velocity ($u-v$) and stress-velocity ($\sigma-v$) forms, in a single weighted-residual framework in which every interface coupling is a target-value choice, and compare them on common Floquet-Bloch measures of dispersion, dissipation, and semi-discrete stability. Three findings stand out. First, no sampled member of the velocity-trace family is stable across the tested orders $p = 1, \dots, 6$: most grow exponentially at every order $p \geq 2$, the one member free of growth through $p = 3$ grows from $p = 4$, and two others fail more weakly, through a defective zero eigenvalue giving algebraic growth. An energy identity locates the defect in the kinematic statement, which pairs $\dot{u}-v$ in the L^2 (mass-matrix) inner product and omits the compatibility flux the energy estimate requires. Second, the energy-paired $\sigma-v$ form attains the optimal orders $2p+2$ in phase and $2p+1$ in dissipation and converges to the exact transfer-matrix band structure of a high-contrast layered cell. Third, read at real frequency, the same members exhibit negative spatial attenuation on rightward group-velocity branches. Bisection locates the SIPG coercivity threshold at exactly $p(p+1)/2$ for $p \leq 6$, an exact constant for the classical $\mathcal{O}(p^2)$ law.

Keywords: discontinuous Galerkin, dispersion analysis, Bloch analysis, Riemann solutions, pollution error, heterogeneous media

1. Introduction

The accuracy of a wave propagation simulation over long times depends strongly on the dispersion properties of its spatial discretization, since over the dozens or hundreds of wavelengths that a transient signal traverses, small phase and amplitude errors per wavelength can accumulate into a significant error source. Quantifying how a semi-discretization propagates a plane wave, including its numerical dispersion

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(phase error), dissipation (amplitude error), and the stability of its full discrete spectrum, has therefore been a central tool of method design since the earliest finite element analyses [1–3]. For the continuous Galerkin (CG) method, consistent and lumped mass matrices bracket the exact dispersion relation from above and below for linear elements [4], their leading errors being equal and opposite. On the other hand, spectral element methods recover high-order accuracy with diagonal mass on Gauss–Lobatto nodes [5–7], and discrete-spectrum analyses have driven developments from isogeometric analysis [8–11] to mixed discretizations [12].

Discontinuous Galerkin (DG) [13–18] method offers several advantages for elastodynamics, such as element-wise balance, block-diagonal mass matrices, hp-flexibility, and a natural place for material discontinuities at element boundaries, but it also enlarges the design space. Two largely independent choices define a DG discretization. The first choice is the *set of fields*. The second-order (displacement) form can be discretized directly with interior-penalty (IP) techniques descending from [19–21], in symmetric (SIPG), nonsymmetric (Baumann–Oden and NIPG) [22–24], or incomplete (IIPG) [25] variants, all unified in the framework of [26]; alternatively, the problem can be written as a first-order system in stress–velocity (σ – v) or displacement–velocity (u – v) pairs and discretized with numerical fluxes [18, 27, 28]. The second choice is the *numerical trace*, which may be the upwind flux obtained from the exact Riemann problem, a central average with or without penalty, or the alternating one-sided traces of the local DG (LDG) method [27]. Both choices interact with the time-domain energy balance in ways that elliptic-based intuition does not predict [29–34].

The dispersive and dissipative behavior of DG discretizations has accordingly received sustained attention. For first-order hyperbolic systems, the eigensolution analyses of [35, 36] and the sharp asymptotic results of [37] established the very high orders of $2p + 2$ and $2p + 1$ for the phase and dissipation errors of upwind DG, refined and extended in [38–41]; eigensolution techniques have since become standard tools in the high-order computational fluid dynamics community [42–45]. In the frequency domain the same phase errors drive the pollution effect of Helmholtz discretizations, in which the accumulated phase lag over many wavelengths dominates the error budget [46–48]. For the second-order wave equation, [49] analyzed the IP family and a class of first-order-system schemes for acoustics, and comparisons of CG and DG date back to [50, 51]. In computational seismology, grid-dispersion analyses cover IP variants [52, 53], ADER-DG [54–56], flux-based σ – v DG with different fluxes and elements [57–60], polytopal DG [61], and spectral-element baselines [62, 63]; hybridizable DG has been analyzed in [64]. Spacetime DG formulations [65–72] couple elements through the Riemann values of the elastodynamic system [73, 74], the *target values* of the present paper, and supply the transfer-matrix and spectral-stability formulation [75] that we reuse here.

Three observations motivate this work. First, the studies above analyze essentially one formulation family at a time. The IP variants are compared with each other [52], flux choices are compared within the stress–velocity system [59, 60], and CG is compared with a single DG member [50, 62, 63]; we are not aware of a

study that crosses the two design axes, namely the field set and the numerical trace, in a single comparison. The closest is [49], which analyzes both the IP family on the second-order form and a class of first-order-system schemes for acoustics, and so crosses the field-set axis; it fixes the trace family within each, and addresses neither semi-discrete stability over the Brillouin zone nor heterogeneous media, which are the two axes along which the present comparison separates the formulations. Second, the standard analyses report frequency (dispersion–dissipation) information of selected branches, while semi-discrete *stability* over the full Brillouin zone is taken for granted; as we show, for the mass-paired displacement–velocity formulations analyzed here it cannot be. Third, almost all available results concern homogeneous media. Heterogeneous ones, with their band gaps, effective-medium limits, and realization-to-realization variability [76–83], are where exact references are available in one dimension, transfer matrices for periodic cells and homogenization limits for random ones, and they remain unused as a check on whether a homogeneous-medium ranking survives the impedance contrasts, band gaps, and interface reflections of a real medium.

Three gaps follow. A side-by-side dispersion–dissipation–stability comparison of one-field and two-field DG formulations on identical metrics is missing; the stability of velocity-trace formulations has not been characterized as a function of polynomial order, flux, penalty, and symmetrization, in either the time or the frequency domain; and the accuracy ranking of DG formulations in periodic (phononic) media, against exact transfer-matrix band structures, is not documented. The velocity-trace construction is what the most direct route to a first-order system produces: interpolate u and v , enforce $\dot{u} = v$ in the L^2 sense that the mass matrix supplies, and couple the elements through the same momentum flux the second-order form already uses. It is also the semi-discrete counterpart of our own spacetime DG formulations, in which the equation of motion is enforced in rate form and weighted by the test velocity [68]. We are not aware of its use as a stand-alone spatial discretization, and what follows is accordingly not a critique of a published method but a demonstration that the choice of kinematic pairing, an aspect of the construction that is usually left implicit, decides stability.

Our main contributions can be summarized as follows.

First, we embed CG and four DG formulations of one-dimensional elastodynamics, namely the displacement-trace (u^*) and velocity-trace (v^*) one-field forms and the displacement–velocity (u – v) and stress–velocity (σ – v) two-field forms, in a single weighted-residual framework. All interface coupling is expressed through common target-value families (Riemann, central, alternating; penalty η_I ; symmetrization ε), so the members differ in as few design choices as the formulations permit, and we name the choice explicitly wherever more than one changes at once.

Second, we screen the standard trace, penalty, and symmetrization choices for semi-discrete stability over the Brillouin zone, together with a 17×21 penalty–symmetrization grid scan for the symmetrized velocity-trace member. No sampled member of this family is stable across the tested orders $p = 1, \dots, 6$, though the failure is neither at every order for every member nor always of the same kind. Most fail by exponential

growth at every tested order $p = 2, \dots, 6$; the two-field u - v Riemann/ $\varepsilon = +1$ pairing is free of growth for $p \leq 3$ and grows at $p = 4, 5, 6$; and two members fail more weakly, through a defective zero eigenvalue that yields algebraic rather than exponential growth. We trace the mechanism, for the unsymmetrized members, to the mass rather than energy pairing of the kinematic statement, and show that at real frequency the same members exhibit negative spatial attenuation on rightward group-velocity branches, the frequency-domain Riemann coupling of a single displacement field included. The requirement this exposes is not specific to the members sampled: in a two-field formulation the auxiliary equation must be tested in the inner product that pairs it with the momentum equation in the energy, and an L^2 pairing that appears equally natural forfeits the interface flux the energy estimate needs.

Third, we confirm that the displacement-trace family reproduces classical IP theory, and sharpen one part of it: bisecting on the penalty locates the SIPG coercivity threshold at exactly $p(p+1)/2$ for $p = 1, \dots, 6$, giving the classical $\mathcal{O}(p^2)$ law an exact rational constant at the six orders tested, which we conjecture holds at every order; and the same screening exposes a wave-specific instability of the Baumann–Oden member at $p = 3$.

Fourth, we measure phase and dissipation accuracy on common metrics. The energy-paired stress–velocity formulation attains the optimal orders, $2p+2$ in phase and $2p+1$ in dissipation with Riemann targets and a conservative $2p+2$ with alternating targets, and converges to the exact phononic band structure of a high-contrast layered cell with higher asymptotic order and, on refined meshes, substantially lower error than the nonsymmetric IP members. The per-degree-of-freedom comparison is more nuanced, since the two-field members carry twice the unknowns. The one displacement–velocity member free of exponential growth at low order, by contrast, dissipates resolved waves at first order.

The remainder of this paper is organized as follows. §2 presents the unified weak formulation, its five members, and their energy structure; §3 the Bloch dispersion–dissipation–stability metrics and the exact transfer-matrix reference; §4 the stability map; §5 the homogeneous-medium accuracy; §6 its frequency-domain (Helmholtz) reading; §7 the layered-media results; and §8 the conclusions. Appendix A lists the parent element matrices, Appendix B details the two-dimensional spot check, and Appendix C the higher-order stability scans.

2. A unified family of weak formulations

2.1. Model problem

We consider the linear elastodynamics of a one-dimensional bar of finite length,

$$\rho(x) \dot{v} = \frac{\partial \sigma}{\partial x}, \tag{1a}$$

$$\sigma = E(x) \frac{\partial u}{\partial x}, \tag{1b}$$

in which u is the displacement, $v = \dot{u}$ the velocity, σ the stress, ρ the density, and E the elastic modulus; superposed dots denote time derivatives and body force is omitted as it plays no role in dispersion analysis. The local wave speed and impedance are $c = \sqrt{E/\rho}$ and $Z = \rho c = \sqrt{E\rho}$. Equation (1a) is the balance of linear momentum (BLM) and (1b) is the constitutive relation. Eliminating σ yields the second-order form $\rho \ddot{u} = (Eu_{,x})_{,x}$, whereas retaining (σ, v) or (u, v) as independent unknowns yields first-order systems; the formulations compared in this work differ precisely in which of these forms is discretized and in how inter-element continuity is enforced.

The domain is partitioned into elements $e = 1, \dots, n$ with sizes h_e , and all fields are approximated by nodal Lagrange polynomials of order p within each element; the systematic comparison uses $p \leq 3$, while the single-cell stability scans of §4 extend to $p = 6$ with exact rational element matrices (with symbolically exact fraction arithmetic). For the DG formulations continuity is not built into the approximation space and each element carries its own degrees of freedom and the fields are double-valued at interfaces, with continuity recovered only weakly through the target values below; whereas for CG the end nodes are shared and the space is C^0 by construction. At an interface between elements L (left) and R (right) the traces of a field f are denoted f^L and f^R . We define the average and jump operators

$$\{f\} = \frac{1}{2} (f^L + f^R), \quad \llbracket f \rrbracket = f^R - f^L. \quad (2)$$

The outward unit normal of the left element at the interface is $n = +1$ and that of the right element is $n = -1$.

2.2. Interface target values

Following the spacetime DG literature of our group [68], the coupling between elements is expressed through *target values* σ^* and w^* , single-valued replacements for the stress and for the primary kinematic field w on the interface; $w = v$ for the velocity-trace formulations and $w = u$ for the displacement-trace formulation introduced below. All target values considered here are linear combinations of the four interface traces,

$$\sigma^* = a_1 \sigma^L + a_2 \sigma^R + a_3 w^L + a_4 w^R, \quad w^* = b_1 \sigma^L + b_2 \sigma^R + b_3 w^L + b_4 w^R, \quad (3)$$

and Table 1 lists the three families compared in this work.

The *Riemann* target values are the exact solution of the Riemann problem of (1) at the interface [68, 73],

$$\sigma^* = \frac{Z_R \sigma^L + Z_L \sigma^R}{Z_L + Z_R} + \frac{Z_L Z_R}{Z_L + Z_R} \llbracket v \rrbracket, \quad (4a)$$

$$v^* = \frac{Z_L v^L + Z_R v^R}{Z_L + Z_R} + \frac{\llbracket \sigma \rrbracket}{Z_L + Z_R}, \quad (4b)$$

which reduce to $\sigma^* = \{\sigma\} + \frac{1}{2}Z\llbracket v \rrbracket$ and $v^* = \{v\} + \llbracket \sigma \rrbracket / (2Z)$ for a homogeneous medium. They are the characteristic (upwind) fluxes of the first-order system and render the interface coupling dissipative, penalizing jumps in the traces with the physical impedance.

The *central* target values are the plain averages $\sigma^* = \{\sigma\}$, $w^* = \{w\}$, optionally augmented by a penalty term on the stress target,

$$\sigma^* \leftarrow \sigma^* + \eta \llbracket w \rrbracket, \quad \eta = \begin{cases} \eta_{\text{I}} \frac{Z_{\text{L}} Z_{\text{R}}}{Z_{\text{L}} + Z_{\text{R}}} & w = v, \\ \eta_{\text{I}} \frac{\tilde{E}}{\tilde{h}} & w = u, \end{cases} \quad (5)$$

in which $\tilde{E}$ and $\tilde{h}$ are the arithmetic means of the moduli and sizes of the two elements and $\eta_{\text{I}} \geq 0$ is a dimensionless penalty factor. The impedance scaling for $w = v$ mirrors the velocity-jump term of (4a), so that the Riemann values are recovered exactly from the central family in a homogeneous medium when $\eta_{\text{I}} = 1$ and the $\llbracket \sigma \rrbracket$ term of (4b) is omitted; the $\tilde{E}/\tilde{h}$ scaling for $w = u$ is the classical IP choice.

The *alternating* target values take the stress and kinematic targets from opposite sides, $(\sigma^*, w^*) = (\sigma^{\text{L}}, w^{\text{R}})$ or $(\sigma^{\text{R}}, w^{\text{L}})$, again with the penalty of (5); they are the one-dimensional analog of the alternating traces of the LDG method [27, 28].

Table 1: Target-value families.		
family	σ^*	w^*
Riemann	$\{\sigma\} + \frac{1}{2}Z\llbracket v \rrbracket$	$\{v\} + \llbracket \sigma \rrbracket / (2Z)$
central	$\{\sigma\} + \eta \llbracket w \rrbracket$	$\{w\}$
alternating	$\sigma^{\text{L}} + \eta \llbracket w \rrbracket$ or $\sigma^{\text{R}} + \eta \llbracket w \rrbracket$	w^{R} or w^{L}

2.3. The four DG formulations and CG

All members enforce the BLM (1a) element-wise through the same weighted-residual statement, in which the discrete solution satisfies, for every element e and every test function $\hat{w}$,

$$\int_e \hat{w} \rho \dot{v} \, dx + \int_e \hat{w}_{,x} \sigma \, dx - [\hat{w} \sigma^* n]_{\partial e} = 0, \quad (6)$$

in which $[g]_{\partial e}$ denotes the sum of g over the two element ends, and the boundary flux of the integration by parts is replaced by the target value σ^* . The members differ in (i) the set of independent fields, (ii) the kinematic statement that complements (6), and (iii) the field that plays the role of w in the target values.

2.3.1. One-field, displacement trace (u^*)

The constitutive relation (1b) is substituted into (6), $v = \dot{u}$, $w = u$, and the test function is $\hat{w} = \hat{u}$. Inter-element compatibility is enforced weakly by the adjoint (symmetrization) term, and the complete interface

contribution of an element end to the residual reads

$$-\hat{u} \sigma^* n - \varepsilon \hat{\sigma} n (u^* - u), \quad \hat{\sigma} = E \hat{u}_{,x}, \quad (7)$$

with $\varepsilon \in \{-1, 0, +1\}$. With central targets, (7) generates the classical IP family for the second-order wave equation, with $\varepsilon = -1$ the SIPG [21, 29], $\varepsilon = 0$ the IIPG [25], and $\varepsilon = +1$ the nonsymmetric variants, Baumann–Oden for $\eta_I = 0$ [23] and NIPG for $\eta_I > 0$ [24]. The semi-discrete system is

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{0}, \quad (8)$$

with the standard mass matrix and a stiffness matrix that collects the bulk and interface terms.

2.3.2. One-field, velocity trace (v^*)

The same one-field substitution is used, but the BLM is weighted by the test *velocity* and the targets act on $w = v = \dot{u}$, so that a velocity-jump penalty represents physical impedance-weighted dissipation rather than a stiffness-like penalty. The single unknown field is still u ; $\hat{v}$ and $\hat{u}$ denote members of the same element space, and no time derivative of a test function is implied. This mirrors our spacetime DG formulations, in which the equation of motion is enforced in rate form and weighted by the test velocity [68]. Because v -dependent interface terms multiply $\dot{\mathbf{a}}$, they assemble into a damping matrix and the semi-discrete system becomes

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{0}; \quad (9)$$

in particular the Riemann velocity-jump term of (4a) renders $\mathbf{C}$ positive semi-definite. The adjoint term, $-\varepsilon \beta_e \hat{\sigma} n (v^* - v)$ with $\hat{\sigma} = E \hat{u}_{,x}$, carries the element time scale $\beta_e = h_e/c_e$ (the mean of the two neighboring elements is used on interior interfaces), which compensates the one-time-derivative mismatch between the velocity weighting of (6) and the displacement-based test stress.

2.3.3. Two-field, displacement–velocity (u – v)

Displacement and velocity are interpolated independently, and the BLM (6) is tested with $\hat{v}$ and $w = v$. The kinematic relation $\dot{u} = v$ is enforced by testing its residual against $\hat{u}$ in the unweighted L^2 inner product of the element, $(\hat{u}, \dot{u} - v)_e = \int_e \hat{u} (\dot{u} - v) dx$, whose matrix representation is the mass matrix $\mathbf{m}$ of Appendix A; we refer to this as the *mass pairing* throughout, in contrast to the energy pairing of the σ – v form below. The weight $\alpha_e = (c_e/h_e)^2$ renders the kinematic row dimensionally commensurate with (6), a unit reference density being implied by the nondimensionalization; for $\varepsilon = 0$ it merely rescales those rows and does not affect the discrete spectrum, whereas for $\varepsilon \neq 0$ it sets the relative weight of the adjoint term. The kinematic statement reads

$$\alpha_e \left[\int_e \hat{u} \dot{u} dx - \int_e \hat{u} v dx \right] - \varepsilon [\hat{\sigma} n (v^* - v)]_{\partial e} = 0, \quad \hat{\sigma} = E \hat{u}_{,x}. \quad (10)$$

The stress in (6) and in the targets is computed from the displacement, $\sigma = Eu_{,x}$. The system is first order in time, $\mathbf{M}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{0}$ with $\mathbf{a}$ collecting (u, v) degrees of freedom. Two features of the mass pairing drive the results. Its integrand carries no derivative of $\hat{u}$, so no integration by parts is performed and, at $\varepsilon = 0$, (10) carries no interface term at all: the kinematic statement is element-local, whereas (6) couples elements through σ^* . And its weight is the scalar α_e , not the material weight $1/E$ with which the σ - v form (11) enforces the same physical relation and which, paired with the ρ weight of (6), makes $\frac{1}{2} \int (\sigma^2/E + \rho v^2)$ the natural norm of that system. §2.4 identifies the first feature as the loss of precisely the $\llbracket \sigma \rrbracket$ interface flux that the energy estimate requires, and §4 shows that this choice governs the stability of the entire family.

2.3.4. Two-field, stress-velocity (σ - v)

Stress and velocity are interpolated independently, so the stress equation carries its own test function $\hat{\tau}$, a test stress free in the element space rather than the derived $\hat{\sigma} = E\hat{u}_{,x}$ of the displacement-based members. The constitutive relation is enforced in rate form by testing the residual of $\dot{\sigma} = Ev_{,x}$ against $\hat{\tau}$ in the compliance-weighted inner product $\int_e \hat{\tau}(\cdot)/E \, dx$, whose matrix representation is the weighted mass block $(1/E)\mathbf{m}$. We call this the *energy pairing*: together with the ρ weight that (6) places on $\dot{v}$, it makes the mechanical energy $\frac{1}{2} \int_e (\sigma^2/E + \rho v^2) \, dx$ the natural norm of the discrete system, as §2.4 makes precise. The statement reads

$$\int_e \hat{\tau} \frac{\dot{\sigma}}{E} \, dx + \int_e \hat{\tau}_{,x} v \, dx - [\hat{\tau} v^* n]_{\partial e} = 0, \quad (11)$$

together with (6) tested by $\hat{v}$ ($w = v$). Both equations carry their own target value, so no adjoint term (and no ε) arises; (11) is the weak statement of $\dot{\sigma} = Ev_{,x}$ with its boundary flux replaced by v^* , exactly mirroring how (6) treats σ^* . In contrast to the mass pairing of (10), the integrand here carries the derivative $\hat{\tau}_{,x}$, so integration by parts does produce a boundary term, and it is this v^* flux that supplies the $\llbracket \sigma \rrbracket$ contribution absent from the u - v energy balance. This is the classical first-order-system DG for the acoustic/elastic wave equation [18]; with Riemann targets it coincides with upwind-flux DG, whose energy stability at every polynomial order is classical. The system is again $\mathbf{M}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{0}$, now with the energy-weighted mass blocks $(1/E)\mathbf{m}$ and $\rho\mathbf{m}$.

2.3.5. CG

The continuous Galerkin discretization of the second-order form serves as the baseline. The single field is again the displacement, with $v = \dot{u}$ and $\sigma = Eu_{,x}$ as in the u^* member, and (6) is tested with $\hat{u}$. Because the trial and test spaces are C^0 , every trace is single-valued and every jump of (2) vanishes, so the interface contributions of two neighboring elements cancel identically and no target value enters. The weak form reduces to the classical statement

$$\sum_e \left(\int_e \hat{u} \rho \ddot{u} \, dx + \int_e \hat{u}_{,x} E u_{,x} \, dx \right) = 0, \quad (12)$$

with the semi-discrete system (8) and no damping matrix: the baseline is conservative at every order and its spectrum is real, so it registers phase error alone.

Two properties of this baseline recur below. First, an assembled CG cell carries p unknowns, the shared end node being counted once, against $p+1$ per field for a DG cell, so comparisons at equal mesh and at equal number of unknowns are not equivalent and §7 reports both. Second, the mass matrix is used in both the consistent and the row-sum lumped forms of Appendix A, the latter being the standard choice for explicit time stepping, whose accuracy cost §5 quantifies. The interior nodes are equispaced throughout, as for the DG members; Gauss–Lobatto placement, *i.e.*, the spectral element method [5, 6], is a standard remedy for the lumped-mass loss at $p = 3$ and lies outside the present comparison.

2.4. Energy identities and the role of the kinematic pairing

The stability dichotomy documented in §4 can be traced precisely to the energy structure of the kinematic statements, which we formalize here for a periodic mesh (no boundary terms). Throughout, $\mathcal{E}$ denotes the discrete mechanical energy of the formulation at hand and all interface sums use the operators (2).

Proposition 1 (energy rate of the σ – v formulation). *Let (σ_h, v_h) solve (6) and (11) on a periodic mesh and let $\mathcal{E} = \frac{1}{2} \sum_e \int_e (\sigma_h^2/E + \rho v_h^2) dx$. Then*

$$\dot{\mathcal{E}} = \sum_{\text{interfaces}} ([\sigma_h v_h] - \sigma^* [v_h] - v^* [\sigma_h]) = - \sum_{\text{interfaces}} (c_v [v_h]^2 + c_\sigma [\sigma_h]^2), \quad (13)$$

with $(c_v, c_\sigma) = (Z_L Z_R / (Z_L + Z_R), 1/(Z_L + Z_R))$ for Riemann targets and $(\eta, 0)$ for central and alternating targets. In particular $\dot{\mathcal{E}} \leq 0$ for every target family, penalty, polynomial order, and mesh, and $\dot{\mathcal{E}} = 0$ for central and alternating targets with $\eta = 0$.

PROOF. Testing (11) with $\hat{\tau} = \sigma_h$ and (6) with $\hat{w} = v_h$ and summing over elements turns the bulk terms into $\int_e (\sigma_h v_h)_{,x}$ per element, whose boundary sums recombine at each interface into the first equality of (13). Using the product rule $[\sigma v] = \{\sigma\} [v] + [\sigma] \{v\}$ and writing $\sigma^* = \{\sigma\} + a_\sigma$, $v^* = \{v\} + a_v$ reduces the interface sum to $-\sum (a_\sigma [v] + a_v [\sigma])$, and inserting the deviations a_σ , a_v of Table 1 gives the quadratic form. At a bimaterial interface the Riemann deviations (4) carry, beyond the jump terms retained in (13), the impedance-contrast contributions $+c_{\sigma v} [\sigma]$ in a_σ and $-c_{\sigma v} [v]$ in a_v , with

$$c_{\sigma v} = \frac{Z_L - Z_R}{2(Z_L + Z_R)}, \quad |c_{\sigma v}| < \frac{1}{2}. \quad (14)$$

These enter $-\sum (a_\sigma [v] + a_v [\sigma])$ as the equal and opposite cross terms $\mp c_{\sigma v} [\sigma_h] [v_h]$ and cancel identically, so (13) holds at arbitrary impedance contrast.

Proposition 1 is the specialization of classical first-order-system DG energy analysis [18, 31, 33, 84] to the target-value family of Table 1; the conserving/dissipative dichotomy it records is the wave-equation

instance of the general jump-penalty energy estimate for symmetric hyperbolic systems [84]. It serves here as the contrast to Proposition 2 below. Since the energy is a norm of the Bloch amplitude, the σ - v form carrying no displacement state and hence no rigid mode, it implies $\text{Im} \omega \leq 0$ for every Bloch mode of this formulation at every order, under the $e^{-i\omega t}$ convention of §3.1 in which $\text{Im} \omega > 0$ denotes growth, so its unconditional semi-discrete stability is a known property that our spectra merely confirm. The cancellation of the impedance-contrast cross terms recorded in the proof is structural rather than incidental: it occurs because (6) and (11) each carry their own target value, so the two cross terms are generated by the same coefficient $c_{\sigma v}$ with opposite signs. A formulation that supplies a target to only one of the two equations retains the term. The same computation exposes what the velocity-trace formulations lack.

Proposition 2 (energy defect of the velocity-trace formulations). *Let (u_h, v_h) solve the u - v formulation (6), (10) with $\varepsilon = 0$ on a periodic mesh (the v^* one-field formulation with $\varepsilon = 0$ is the special case $v_h \equiv \dot{u}_h$), and let $\mathcal{E} = \frac{1}{2} \sum_e \int_e (E(u_{h,x})^2 + \rho v_h^2) dx$. Then, with $\sigma_h = Eu_{h,x}$ and any stress target of Table 1,*

$$\dot{\mathcal{E}} = - \sum_{\text{interfaces}} (\{\sigma_h\} \llbracket v_h \rrbracket + c_{\sigma v} \llbracket \sigma_h \rrbracket \llbracket v_h \rrbracket + c_v \llbracket v_h \rrbracket^2), \quad (15)$$

in which c_v is the velocity-jump coefficient of Proposition 1 and $c_{\sigma v}$ is given by (14) for Riemann targets, vanishes for the central family, and equals $\mp \frac{1}{2}$ for the two alternating orientations. The last term is dissipative, being upwind dissipation for Riemann targets and penalty dissipation for the central and alternating families, in which $c_v = \eta$ and vanishes at $\eta_I = 0$. The first two are sign-indefinite, and the $\llbracket \sigma_h \rrbracket$ -proportional flux that cancels them in (13) is absent because the kinematic statement (10) carries no interface term. In particular the impedance-contrast term $c_{\sigma v} \llbracket \sigma_h \rrbracket \llbracket v_h \rrbracket$, which cancels identically in the σ - v energy rate, survives here whenever the interface separates materials of unequal impedance.

PROOF. With $\varepsilon = 0$, (10) states that the L^2 projection of $\dot{u}_h - v_h$ onto the element space vanishes; since both fields belong to that space, $\dot{u}_h = v_h$ identically, and the strain-energy rate is $\sum_e \int_e Eu_{h,x} v_{h,x}$ without any boundary term. Testing (6) with v_h gives $\frac{d}{dt} \frac{1}{2} \int_e \rho v_h^2 = - \int_e Eu_{h,x} v_{h,x} + [v_h \sigma^* n]_{\partial e}$ per element. The bulk terms cancel in the sum and the interface terms recombine into $-\sigma^* \llbracket v_h \rrbracket$ per interface. Writing $\sigma^* = \{\sigma_h\} + a_\sigma$ as in Proposition 1 and inserting the deviation $a_\sigma = c_{\sigma v} \llbracket \sigma_h \rrbracket + c_v \llbracket v_h \rrbracket$ of the target at hand yields (15). Unlike in Proposition 1, no second interface term is available to cancel the $c_{\sigma v}$ contribution, since (10) supplies no target value.

Proposition 2 makes the instability mechanism quantitative. The identities (13) and (15) are stated for real time-domain fields; evaluated on a complex Bloch mode (§3.1) under the $e^{-i\omega t}$ convention, the quadratic quantities are interpreted in the sesquilinear sense, $\mathcal{E} = \frac{1}{2} \sum_e \int_e (E|u_{h,x}|^2 + \rho|v_h|^2) dx$ and $fg \mapsto \text{Re}(f\bar{g})$ in the interface products, under which $\dot{\mathcal{E}} = 2 \text{Im} \omega \mathcal{E}$. The growth rate therefore equals the right side of (15) divided by $2\mathcal{E}$, and the mode grows whenever the indefinite term injects more energy than the impedance

term removes. Evaluating both terms on the computed most-unstable mode of the u - v Riemann/ $\varepsilon=0$ member at $p = 2$ ($kh \approx \pi$), normalized by $2\mathcal{E}$, gives an injection of $+0.441$ against a dissipation of -0.071 in units of c/h , whose sum reproduces the eigenvalue growth rate $\text{Im}\omega h/c = +0.370$ to the eigensolver tolerance. Figure 1 traces this balance over the Brillouin zone, plotting both terms of (15), normalized by $2\mathcal{E}$, on the least stable propagating Bloch mode of each wavenumber. At $p = 1$ the indefinite term is itself nonpositive along this branch, so the mode decays, which is the first-order dissipation measured in §5, whereas for $p \geq 2$ the injection turns positive and prevails over part of the zone, and the sum, which equals the growth rate $\text{Im}\omega$, rises above zero. Proposition 2 thus shows that the $p = 1$ neutrality is not energetically protected, and the computed balance shows how it is lost at higher orders. What the richer trace space does to the correlation between $\{\sigma_h\}$ and $\llbracket v_h \rrbracket$ along Bloch branches, and hence why the sign changes at $p = 2$, we have not isolated; settling it would require a mode-content analysis that we do not undertake.

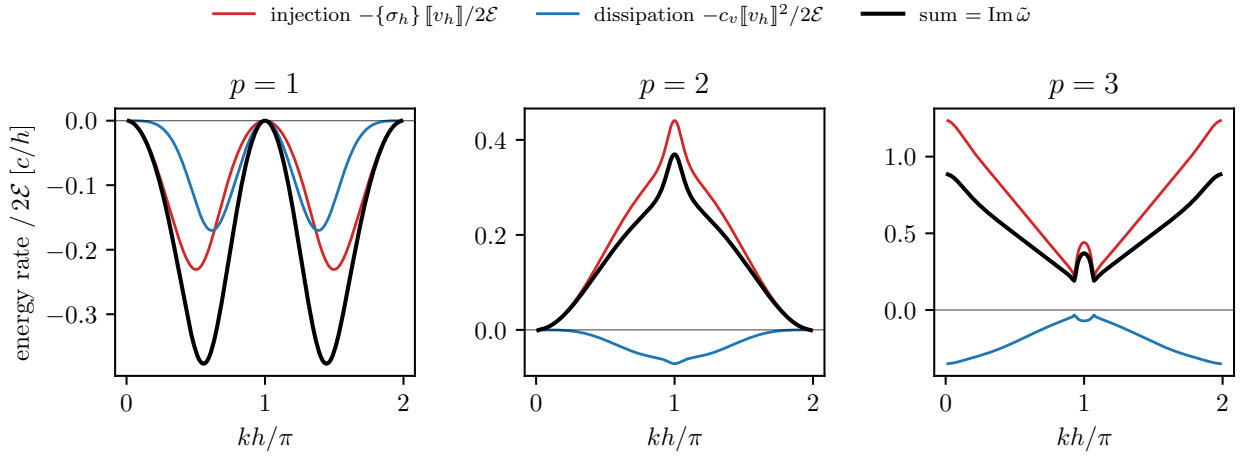


Figure 1: Energy-rate decomposition (15) of the least stable propagating Bloch mode of the u - v Riemann/ $\varepsilon = 0$ member, normalized by $2\mathcal{E}$, over the Brillouin zone for $p = 1, 2, 3$.

Proposition 2 is derived for $\varepsilon = 0$, and we do not extend it. For $\varepsilon \neq 0$ the kinematic statement no longer enforces $\dot{u}_h = v_h$ exactly, the added term contributes a mass-inverse lifting to the energy balance, and we have not derived the corresponding identity. The $\varepsilon \neq 0$ results reported below are therefore *numerically consistent with* the missing-flux mechanism of (15) rather than predicted by it. Two distinct scalars govern that term and should not be conflated: the one-field v^* member carries the element time scale $\beta_e = h_e/c_e$ of its adjoint term, whereas the u - v member carries the kinematic weight $\alpha_e = (c_e/h_e)^2$ of (10). Neither removes the defect. Over the sampled penalty-symmetrization grid of §4 the growth of the symmetrized v^* member is positive at all 356 non-degenerate sampled (η_I, β) pairs. Among the u - v members the Riemann/ $\varepsilon = +1$ combination is the only one free of exponential growth at $p \leq 3$, and it too grows at $p = 4, 5, 6$ (Appendix C),

so relative to the $\varepsilon = 0$ sequence the adjoint term delays the observed onset of growth and suppresses its rate without repairing the energy structure. Two further members escape exponential growth at some orders by a different route, carrying a defective $\omega = 0$ eigenvalue instead, and §4 treats them separately; neither is stable.

Testing the kinematic residual in the strain-energy inner product instead of the mass inner product supplies exactly the missing $\llbracket \sigma \rrbracket$ flux. The resulting statement is the rate form of the constitutive equation (11) with σ_h constrained to $Eu_{h,x}$, that is, the energy pairing rather than the L^2 projection.¹ This requirement is implicit throughout the energy-based and spacetime DG literature, in which the auxiliary velocity equation is invariably coupled through an energy-consistent numerical flux [33, 68] rather than an L^2 projection; the same mass pairing is, by contrast, stable and optimal for the *time*-discontinuous Galerkin formulation [65], in which no spatial jump structure is present. We are not aware of a previous statement of the energy defect (15) or of the resulting loss of stability of the mass-paired u - v spatial discretization over the tested orders $p \leq 6$, and they make explicit the role of the kinematic pairing on which those constructions rely tacitly. We also note that the integration-by-parts structure behind Propositions 1 and 2 is not specific to one dimension, edge integrals taking the place of interface values; §4 reports a two-dimensional spot check for the *scalar* wave operator.²

All formulations are implemented in a single in-house 1D code (see the data-availability statement) with hardcoded nodal Lagrange elements of order $p \leq 3$; those parent element matrices, including the mixed gradient–value matrix required by (11), are listed in Appendix A. The single-cell tools that carry the scans to higher order, and the checks run among them, are described in §3.4.

3. Bloch dispersion, dissipation, and stability framework

3.1. Bloch eigenproblem

All formulations of §2 lead to semi-discrete systems of the forms (8), (9), or $\mathbf{M}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{0}$. To characterize their interior (boundary-independent) behavior we use Floquet–Bloch analysis. The medium is taken periodic with period L , a single element for homogeneous media and the unit cell for layered media, and solutions are sought in Bloch form, so that the degrees of freedom of the cell repeat with the phase factor

$$\gamma = e^{ikL}, \quad (16)$$

in which k is the (real) Bloch wavenumber. Inserting this ansatz couples the two ends of the cell, so that traces of the right neighbor are those of the cell multiplied by γ , and those of the left neighbor by γ^{-1} . For

¹We do not claim an algebraic equivalence of the two discrete systems, whose unknown sets differ.

²The full tensor elastodynamic system is not addressed by Appendix B, and we claim nothing for it here.

CG the shared end node is eliminated by the same relation. The result is a small complex eigenproblem per wavenumber. With the time-harmonic convention

$$\mathbf{a}(t) \sim e^{-i\omega t}, \quad (17)$$

the second-order members yield $(-\omega^2 \mathbf{M} + \mathbf{K}(k)) \mathbf{a} = \mathbf{0}$, the damped member (9) the quadratic problem $(-\omega^2 \mathbf{M} - i\omega \mathbf{C}(k) + \mathbf{K}(k)) \mathbf{a} = \mathbf{0}$ (solved in companion form), and the first-order members

$$(-i\omega \mathbf{M} + \mathbf{K}(k)) \mathbf{a} = \mathbf{0}. \quad (18)$$

Each eigenvalue is a complex frequency $\omega(k)$ of a discrete Bloch mode $e^{i(kx - \omega t)}$, in which $\text{Re } \omega$ is the propagating frequency and $\text{Im } \omega$ the growth rate, with $\text{Im } \omega < 0$ denoting dissipation and $\text{Im } \omega > 0$ exponential growth, *i.e.*, an unstable semi-discretization. Three properties of each formulation follow directly. The first is *stability*, the requirement $\max_k \max_j \text{Im } \omega_j(k) \leq 0$. The second is the *dispersion error*, the deviation of $\text{Re } \omega$ of the physical (acoustic) branch from the exact $\omega = ck$ in homogeneous media or from the exact Bloch frequency in heterogeneous media. The third is the *dissipation error*, $-\text{Im } \omega$ of the physical branch, which for a conservative target family is identically zero. For homogeneous media we report the dimensionless frequency $\tilde{\omega} = \omega h/c$ as a function of kh , identify the physical branch as the eigenvalue closest to ck in the complex plane, and measure the relative phase and dissipation errors

$$e_{\text{ph}} = \frac{|\text{Re } \omega - ck|}{ck}, \quad e_{\text{ds}} = \frac{|\text{Im } \omega|}{ck}. \quad (19)$$

Because the semi-discrete operator has real coefficients, conjugating a Bloch mode sends k to $-k$ and ω to $-\bar{\omega}$, so that

$$\omega(-k) = -\overline{\omega(k)} \quad (20)$$

for every member and every target family. At a *fixed* wavenumber this becomes a symmetry of the spectrum only when the member is additionally invariant under spatial reflection. The two-sided Riemann and central targets are, and their spectra are therefore self-paired under $\omega \mapsto -\bar{\omega}$, which preserves $\text{Im } \omega$ and so pairs each forward-traveling mode with a backward-traveling one of the same growth or decay rate. The one-sided alternating targets are not: reflection exchanges the two orientations, and for the velocity-trace members the spectrum at fixed k is correspondingly not self-paired. For the σ - v alternating member the distinction is invisible, its $\eta_1 = 0$ spectrum being real and hence symmetric under $\omega \mapsto -\omega$ in any case.

The mode count is unaffected by these distinctions. Writing n_f for the number of interpolated fields, a DG cell carries $p+1$ unknowns per field and yields $2(p+1)$ frequencies per wavenumber, either as $\pm$ counterparts of a second-order problem ($n_f = 1$) or as the roots of a first-order problem in $2(p+1)$ unknowns ($n_f = 2$), while CG, whose end nodes the Bloch relation identifies, yields $2p$. Two of these frequencies constitute the physical acoustic branch, one for each propagation direction. The remaining $2p$ for the DG members, and $2(p-1)$ for CG, are nonphysical, and their location in the complex plane, whether harmlessly damped or unstable, is part of the comparison.

3.2. Frequency-domain dispersion: the inverse eigenproblem

Because the interface coupling reaches only nearest neighbors, only the powers γ^{-1} , γ^0 , and γ of the propagation factor $\gamma = e^{ikL}$ appear in the Bloch symbol of any member, which is therefore a Laurent polynomial of degree one in γ ; written for the homogeneous one-element cell in which $L = h$,

$$\mathbf{S}(\gamma; \omega) = \mathbf{S}_0(\omega) + \gamma \mathbf{S}_+ + \gamma^{-1} \mathbf{S}_-, \quad (21)$$

in which $\mathbf{S} = -\omega^2 \mathbf{M} - i\omega \mathbf{C}(\gamma) + \mathbf{K}(\gamma)$ for the second-order members and $\mathbf{S} = -i\omega \mathbf{M} + \mathbf{K}(\gamma)$ for the first-order ones. At a prescribed real frequency, multiplying (21) by γ clears the negative power and leaves a matrix polynomial of degree two, so $\det \mathbf{S} = 0$ is a quadratic eigenproblem in γ . Its linearization carries $2n$ generalized eigenvalues, of which some may be zero or infinite depending on the ranks of the neighbor blocks $\mathbf{S}_\pm$; those ranks vary with the member and the order, from full rank for u^* at $p = 1$ down to unity for the two-field members. The scans below retain all finite, nonzero admissible roots. For each such root $k_h(\omega) = -i \ln(\gamma)/h$ delivers a complex discrete wavenumber, whose real part $\text{Re } k_h$ carries the phase (pollution-driving) error and whose imaginary part gives the spatial attenuation; a negative attenuation on a forward-transporting branch is anomalous.

Branches are classified by the sign of their discrete group velocity, obtained from first-order perturbation of (21) as

$$v_g = -\frac{\mathbf{y}^H \mathbf{S}_{,k} \mathbf{a}}{\mathbf{y}^H \mathbf{S}_{,\omega} \mathbf{a}}, \quad (22)$$

in which $\mathbf{a}$ and $\mathbf{y}$ are the right and left eigenvectors of the (non-Hermitian) pencil. The left–right pairing is essential, since the right-eigenvector quotient is exact only for self-adjoint pencils. This classification separates the propagating branches from the evanescent pairs of band gaps. We therefore use the descriptive term *rightward group-velocity branch* throughout, rather than identifying $\text{sign}(v_g)$ with the direction of energy flux.³

Two conventions fix the branch bookkeeping. The principal logarithm is used, $\text{Re } k_h h \in (-\pi, \pi]$, and dispersion curves are tracked by continuation in ω , each frequency selecting the root nearest the previously tracked wavenumber and starting from ω/c at low frequency, which prevents branch switching at band edges. The gain classification is independent of that tracking, since it scans every admissible root at each frequency and filters only by transport direction.

Four filters define admissibility, and a null gain result is only as strong as they are. Roots with $|\gamma| < 10^{-14}$ or non-finite are discarded as the zero and infinite eigenvalues of the linearization. Roots with $|\text{Im } k_h| h > 5$ are discarded as too strongly evanescent to represent transport. Eigenpairs whose left or right eigenvector

³For the non-Hermitian pencils of the dissipative and unstable members that identification is not automatic. Establishing it would require either a discrete energy-flux functional or a forced finite-domain calculation, neither of which we undertake here.

has norm below 10^{-12} , or whose group-velocity denominator falls below 10^{-10} , are discarded as numerically indeterminate. Of the survivors, only those with $\text{Re } v_g > 10^{-3}$ are retained as rightward, and a gain is recorded when $-\text{Im } k_h$ exceeds 10^{-8} . This inverse reading requires no new discretization machinery, as the blocks of (21) are extracted exactly from the assembled cell matrices, and §6 uses it to transfer the entire comparison to the time-harmonic setting.

3.3. Exact dispersion of layered media

For a layered unit cell the exact dispersion relation follows from the classical transfer-matrix method [76, 77]. With the state vector (u, σ) and time-harmonic motion at frequency ω , propagation across a layer of thickness h_j , modulus E_j , and density ρ_j is

$$\mathbf{T}_j(\omega) = \begin{bmatrix} \cos q_j h_j & \sin(q_j h_j)/(E_j q_j) \\ -E_j q_j \sin q_j h_j & \cos q_j h_j \end{bmatrix}, \quad q_j = \frac{\omega}{c_j}, \quad (23)$$

and the cell matrix is the ordered product $\mathbf{T}_{\text{cell}} = \mathbf{T}_{n_\ell} \cdots \mathbf{T}_1$. Since $\det \mathbf{T}_{\text{cell}} = 1$, Bloch periodicity gives the dispersion relation

$$\cos(kL) = \frac{1}{2} \text{tr } \mathbf{T}_{\text{cell}}(\omega), \quad (24)$$

whose solutions $\omega(k)$ are the exact band structure; frequencies with $|\text{tr } \mathbf{T}_{\text{cell}}| > 2$ form the band gaps. We evaluate (24) by scanning and bisection to machine precision, providing the reference for the layered-media comparison.

3.4. Verification

The cell eigenvalues are assembled by more than one route, each comparison isolating a different part of the construction. The first assembly is that of the general solver, which builds the periodic cell through the routines it uses for heterogeneous media, with γ -weighted wrap-around interface terms. The second is a separately written single-cell script whose parent elements are built by quadrature at arbitrary order; it shares no code with the solver, so the comparison tests the interface terms in which the formulations differ. The third builds the parent matrices in exact rational arithmetic and diagonalizes the pencils at arbitrary working precision, which the stability tables require because roundoff at a defective eigenvalue is indistinguishable from growth in double precision (§3.5); it produces every entry of Tables 2 and C.7 by the procedure of Appendix C, and carries the scans past the order $p \leq 3$ hardcoded in the solver. It admits a modal Legendre basis in place of the nodal Lagrange basis of the same polynomial space, which varies the element construction and its trace evaluation while leaving the interface assembly fixed; that comparison is therefore a basis-invariance check on the elements rather than a second check of the interface terms.

The solver and the quadrature-built script agree to 10^{-12} in every eigenvalue for all formulation/target combinations, the exact-arithmetic assembly reproduces the script to ten digits in double precision, and

its nodal and modal bases agree to a relative 10^{-25} . Two closed forms serve as external references. The CG branch reproduces the lattice dispersion of linear elements exactly, and the σ - v Riemann bands of the layered cell of §7 match (24) to relative errors of 1.6×10^{-7} at $p = 2$ and 6.8×10^{-11} at $p = 3$, both with sixteen elements per layer.

3.5. Classification of stability verdicts

A *defective* eigenvalue, one whose eigenvectors have collapsed so that its largest Jordan block has size $m > 1$, contributes growth t^{m-1} to the solution. A spectrum with $\text{Im } \omega = 0$ at every wavenumber therefore does not by itself imply stability, and §4 reports two members for which it does not.

For a simple eigenvalue the computed $\text{Im } \omega$ settles at roughly $10^{-8} \max_j |\omega_j|$ in double precision, far enough below the dissipation rates of §5 to separate the conservative members from the dissipative ones. At a defective eigenvalue there is no such floor, and the same m that makes the mode grow is what prevents double precision from measuring it. Perturbing an m -fold defective eigenvalue by ϵ splits it into m eigenvalues at distance $\epsilon^{1/m}$, some of which acquire a positive imaginary part; roundoff is such a perturbation, so a computed rate at a defective eigenvalue carries no information about whether it is real. The blocks encountered here reach $m = 8$, at which the double-precision floor is 10^{-2} . Every entry of Tables 2 and C.7 is therefore computed at two working precisions and classified by precision invariance, following the procedure of Appendix C. Both raw rates are archived rather than rounded to a verdict.

The structure itself is determined exactly rather than inferred. At the two wavenumbers where the Bloch factor is rational, $\gamma = \pm 1$, the cell matrices are rational, and the Jordan structure of $\omega = 0$ follows from the nullity sequence $d_j = \dim \ker \mathbf{A}^j$ of the semi-discrete generator $\mathbf{A} = -\mathbf{M}^{-1}\mathbf{K}$, for which $\dot{\mathbf{a}} = \mathbf{A} \mathbf{a}$ and $\omega = 0$ is the zero eigenvalue, computed in integer-fraction arithmetic; the number of blocks of size at least j is $d_j - d_{j-1}$. This is a rank computation in exact arithmetic, and the precision-collapse rate is used only to separate roundoff from growth. Both points are checked, since the defect is not always at the center.

One such block is physical, and we refer to it as the *rigid-body pair*: a constant displacement together with a constant velocity is an $\omega = 0$ double root of every displacement-based formulation on a periodic mesh, CG and the continuum included, and appears as a size-two block at $\gamma = +1$ throughout. The σ - v form, having no displacement state, does not carry it. It is excluded from the classification, and anything beyond it is flagged.

4. Stability screening of the formulation families

Before any accuracy comparison is meaningful, each (formulation, target, ϵ , η_I , p) combination must pass the stability test of §3.1. Table 2 reports the verdicts for $p = 1, 2, 3$, from 161 equally spaced wavenumbers over the Brillouin zone $[0, 2\pi/h]$, with growth rates normalized by c/h ($h = L$ for the single-element

homogeneous cell) and classified as in §3.5. Three outcomes are distinguished there: a check mark for a stable member, a cross followed by the order of magnitude of $\max_k \text{Im } \omega h/c$ for exponential growth, and $\dagger_m$ for algebraic growth t^{m-1} from a defective $\omega = 0$ eigenvalue whose largest Jordan block has size m . The σ - v formulation carries no ε . At $\varepsilon = 0$ the kinematic statement enforces $\dot{u}_h = v_h$ exactly, so the u - v and v^* members are the same discrete operator. Their $\varepsilon = 0$ rows are assembled independently and agree, and serve as a cross-check between the two implementations.

Table 2: Semi-discrete stability of every sampled formulation, target family, ε , and η_I combination in a homogeneous medium, at $p = 1, 2, 3$.

formulation, flux	ε	$p = 1$	$p = 2$	$p = 3$
σ - v , Riemann	—	✓	✓	✓
σ - v , central ($\eta=1$)	—	✓	✓	✓
σ - v , alternating ($\eta=0$)	—	✓	✓	✓
u - v , Riemann	-1	$\times 10^1$	$\times 10^2$	$\times 10^3$
u - v , Riemann	0	✓	$\times 10^{-1}$	$\times 10^{-1}$
u - v , Riemann	+1	✓	✓	✓
u - v , central ($\eta=1$)	-1	✓	$\times 10^1$	$\times 10^1$
u - v , central ($\eta=1$)	0	✓	$\times 10^{-1}$	$\times 10^{-1}$
u - v , central ($\eta=1$)	+1	$\times 10^0$	$\times 10^0$	$\times 10^0$
u - v , central ($\eta=0$)	0	$\dagger_2$	$\dagger_4$	$\dagger_4$
u - v , alternating ($\eta=0$)	0	$\times 10^0$	$\times 10^0$	$\times 10^0$
v^* , Riemann	-1	$\times 10^0$	$\times 10^1$	$\times 10^2$
v^* , Riemann	0	✓	$\times 10^{-1}$	$\times 10^{-1}$
v^* , Riemann	+1	✓	$\times 10^{-2}$	$\times 10^{-1}$
v^* , central ($\eta=1$)	-1	$\times 10^0$	$\times 10^1$	$\times 10^2$
v^* , central ($\eta=1$)	0	✓	$\times 10^{-1}$	$\times 10^{-1}$
v^* , central ($\eta=1$)	+1	✓	$\times 10^{-2}$	$\times 10^{-1}$
v^* , central ($\eta=0$)	0	$\dagger_2$	$\dagger_4$	$\dagger_4$
v^* , alternating ($\eta=0$)	0	$\times 10^0$	$\times 10^0$	$\times 10^0$
u^* , Riemann	-1	$\times 10^0$	$\times 10^1$	$\times 10^1$
u^* , Riemann	0	✓	$\times 10^0$	$\times 10^0$
u^* , Riemann	+1	✓	✓	$\times 10^0$
u^* , central ($\eta=1$)	-1	✓	$\times 10^0$	$\times 10^1$
u^* , central ($\eta=1$)	0	✓	$\times 10^0$	$\times 10^0$
u^* , central ($\eta=1$)	+1	✓	✓	✓
u^* , central ($\eta=0$)	+1	✓	✓	$\times 10^0$
u^* , central ($\eta=2p(p+1)$)	-1	✓	✓	✓
u^* , central ($\eta=2p(p+1)$)	0	✓	✓	✓
u^* , alternating ($\eta=2p(p+1)$)	-1	✓	✓	✓

The stress-velocity family is unconditionally stable. The σ - v formulation is stable for every target family, penalty, and order tested. Riemann targets are energy dissipative, with a non-increasing discrete energy, central and alternating targets with $\eta_I = 0$ are exactly conservative, and any $\eta_I > 0$ adds pure dissipation. This is the content of Proposition 1, and the computed spectra realize it to the eigensolver noise floor.

No sampled velocity-trace member is stable across $p = 1, \dots, 6$. Every sampled combination of the standard trace, penalty, and symmetrization choices fails at one or more of the tested orders, but in two distinct ways,

and the distinction matters. Most fail by exponential growth: they possess Bloch modes with $\text{Im}\omega > 0$, and most v^* and $u-v$ combinations do so at every tested order $p = 2, \dots, 6$. The alternating orientations fail already at $p = 1$, where both one-sided choices give the same normalized rate 1.22, rising to 2.00 and 2.68 at $p = 2, 3$ and growing through $p = 6$ (Appendix C). Two members instead fail *algebraically*, carrying no growing Bloch mode but a defective $\omega = 0$ eigenvalue, and are marked $\dagger_m$ in Table 2. Three members depart from this pattern: $u-v$ Riemann/ $\varepsilon = +1$ shows no growth for $p \leq 3$ but is exponentially unstable at $p = 4, 5, 6$; $u-v$ central/ $\eta_{\text{I}} = 0/\varepsilon = +1$ is exponentially unstable at $p = 1, 2, 3$ and only defective at $p = 4, 5, 6$; and central/ $\eta_{\text{I}} = 0/\varepsilon = 0$ is defective at every tested order and exponentially unstable at none.

The rates are large and, along a fixed sequence, strengthen with order. The Riemann/ $\varepsilon = 0$ member reaches a normalized rate of 0.37 at $p = 2$ and 0.89 at $p = 3$, then 1.56, 2.46, and 3.58 at $p = 4, 5, 6$ in the single-cell analysis of §3.4, whose exact rational integration elements extend beyond the solver’s $p \leq 3$; for $\varepsilon = -1$ the rate reaches $\mathcal{O}(10^3)$ by $p = 3$, and that member is unstable already at $p = 1$. Across the family, however, growth is *not* monotone in p . Appendix C scans the sampled velocity-trace members, namely Riemann and central targets at $\varepsilon \in \{-1, 0, +1\}$ and $\eta_{\text{I}} \in \{0, 1\}$ together with the two alternating orientations at $\varepsilon = 0$, and finds one member losing its exponential growth above $p = 3$ while another acquires it. Monotone strengthening should therefore be asserted only along the Riemann/ $\varepsilon = 0$ and central/ $\varepsilon \in \{-1, 0\}$ sequences in which it is observed, not across the family. Figure 4 traces the growth over the Brillouin zone: at $p = 1$ the spectrum is clean, whereas at $p = 2, 3$ unstable lobes appear and strengthen with order, the worst case approaching the zone center at $p = 3$.

Two members fail algebraically rather than exponentially. Two of the sampled members fail by the defective-eigenvalue mechanism of §3.5 rather than by exponential growth, and their behavior is opposite. The $u-v$ central/ $\eta_{\text{I}} = 0/\varepsilon = +1$ member carries one extra pair beyond the rigid-body one, of size two at every order, so its algebraic growth is linear in t and does not worsen with refinement. The central/ $\eta_{\text{I}} = 0/\varepsilon = 0$ member is the more serious case, and the only sampled member exponentially unstable at no tested order at all. Its blocks grow with the polynomial degree, reaching sizes 2, 4, 4, 6, 6, 8 over $p = 1, \dots, 6$, so the algebraic growth it produces gets *worse* as the discretization is refined in order.

Figure 2 makes the distinction visible, plotting the norm of the semi-discrete solution operator $\exp(\mathbf{A}t)$ at the zone edge. This is the quantity that separates the three outcomes, and the spectrum is not: the defective member has $\text{Im}\omega = 0$ at every wavenumber, so it is indistinguishable from a conservative one on any of the frequency plots in this paper, yet its response grows without bound. Panel (a) shows that growth order by order, and its measured log-log slopes are 1, 3, 5, 7 at $p = 1, 2, 4, 6$, exactly the $m - 1$ predicted by the Jordan blocks of Table 3, which are obtained separately by counting the ranks of the powers of the cell matrix in integer arithmetic. Panel (b) sets the three outcomes side by side at $p = 2$, where the conservative $\sigma-v$ member is flat, the upwind $\sigma-v$ member decays, the defective member rises on a straight line of slope

three, and the exponentially unstable member leaves the frame.

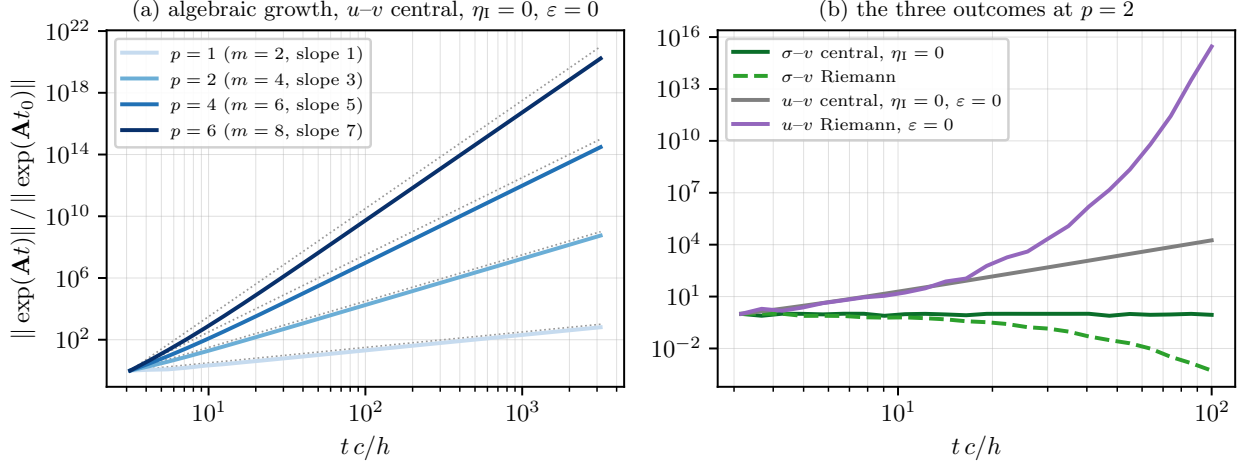


Figure 2: Norm of the semi-discrete solution operator $\exp(\mathbf{A}t)$ at the zone edge, normalized by its value at the first sampled time. (a) The defective central/ $\eta_I = 0/\epsilon = 0$ member at $p = 1, 2, 4, 6$, with reference slopes dotted. (b) The three outcomes at $p = 2$.

Table 3 lists both sequences at the two rational Bloch points, blocks largest first, with the size-two block at $\gamma = +1$ being the rigid-body pair common to all displacement-based formulations. Since the $u-v$ and v^* forms coincide at $\epsilon = 0$, one row covers both. The structure is listed at every order, but it is the operative failure mode only where the member has no exponential growth: for the $\epsilon = 0$ member that is every order, whereas the $\epsilon = +1$ member is exponentially unstable at $p \leq 3$ and defective only from $p = 4$.

Table 3: Exact Jordan structure of the eigenvalue $\omega = 0$ for the two defective velocity-trace members, at the rational Bloch points $\gamma = \pm 1$.

member	γ	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$u-v/v^*$ central, $\eta_I=0, \epsilon=0$	+1	(2, 2)	(2, 2)	(4, 2)	(4, 2)	(6, 2)	(6, 2)
	-1	(2)	(4)	(4)	(6)	(6)	(8)
$u-v$ central, $\eta_I=0, \epsilon=+1$	+1	(2, 2)	(2, 2)	(2, 2)	(2, 2)	(2, 2)	(2, 2)
	-1	(2)	(2)	(2)	(2)	(2)	(2)

The failure is an *interior* property, the periodic-cell analysis containing no boundary, and it is invisible to frequency-only spectrum plots, on which the affected branches appear merely inaccurate rather than unstable. For the $\epsilon = 0$ members its mechanism is the energy defect of Proposition 2: the mass pairing of the kinematic statement supplies no $\llbracket \sigma \rrbracket$ interface flux, the term $-\sum \{\sigma_h\} \llbracket v_h \rrbracket$ in (15) is sign-indefinite, and on the unstable modes it injects energy faster than the impedance term removes it, reproducing the computed growth rate on the mode checked in §2.4 and the sign of the balance over the zone in fig. 1. The same identity explains why the algebraic failures occur where they do: with $\eta_I = 0$ the central family contributes neither the impedance dissipation c_v nor the contrast term $c_{\sigma v}$, leaving the indefinite term alone,

so the spectrum is symmetric about the real axis and the member can only be neutrally stable or defective. For $\varepsilon \neq 0$ we have not derived the corresponding identity, and the observed growth is numerically consistent with the same missing-flux mechanism rather than explained by it. The symmetrization weakens the growth without removing it, which is what one would expect of a term that modifies the kinematic statement by a mass-inverse lifting rather than by supplying the missing energy-paired flux; fig. 3 quantifies that below.

Because Table 2 samples only standard parameter choices, we scan the growth of the symmetrized v^* member over the whole penalty-symmetrization plane $(\eta_I, \beta) \in [0, 8] \times [0, 20 h/c]$, on a uniform 17×21 grid ($\Delta\eta_I = 0.5$, $\Delta\beta = h/c$). Every one of the 356 sampled pairs with $\beta > 0$ grows exponentially, at a rate of at least $4.6 \times 10^{-3} c/h$ at $p = 2$ and 7.4×10^{-3} at $p = 3$. Figure 3 shows how those rates depend on the two parameters, and the dependence is almost entirely on β : the penalty separates the curves only at small β and is irrelevant beyond $\beta \approx 10 h/c$, while the growth falls off as $1/\beta$ and stays positive over the whole sampled range, with no sign of a crossing. Strengthening the symmetrization therefore slows the instability without removing it. The 357th pair is the origin, where the symmetrization vanishes and the member reduces to the central/ $\eta_I = 0/\varepsilon = 0$ discretization discussed above; it does not stabilize the formulation either, failing instead through the defect of Table 3. The whole sampled plane is therefore accounted for and none of it is stable, though a finite grid is not a continuous exclusion.

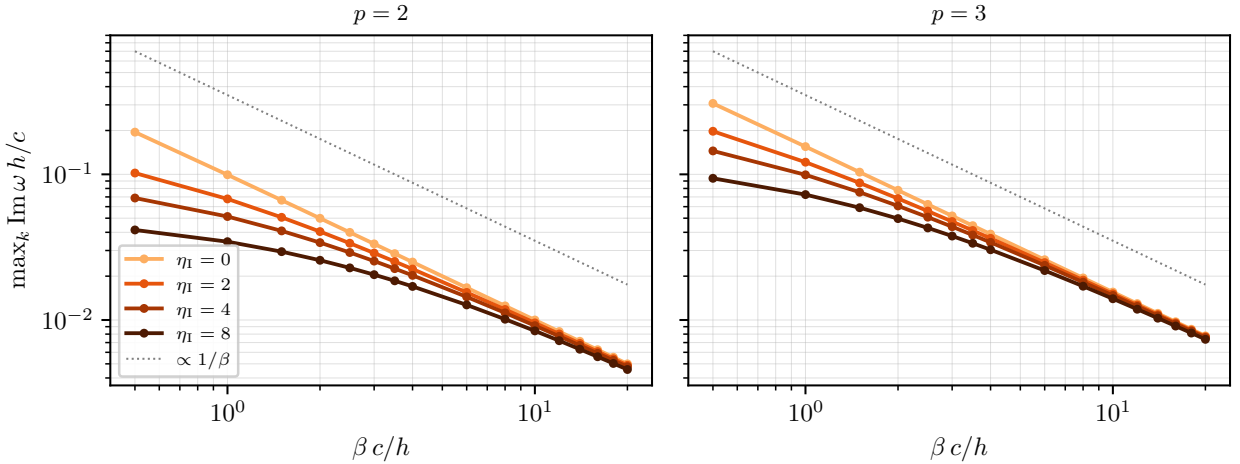


Figure 3: Maximal Bloch growth rate $\max_k \text{Im } \omega h/c$ of the symmetrized v^* central member against the symmetrization time scale β , at four penalties, for $p = 2$ (left) and $p = 3$ (right).

Because the growth is a property of the semi-discrete operator, no consistent time integrator removes it in the limit of small steps. Any convergent one-step method has amplification $R(\Delta t \lambda)$ with $R(z) = e^z + \mathcal{O}(z^{q+1})$, in which $\lambda = -i\omega$ is the eigenvalue of the semi-discrete operator, so unstable semi-discrete modes are amplified for all sufficiently small time steps. At finite Δt the question is no longer asymptotic, and we check it directly for one case: for the most unstable $p = 2$ mode of the u - v Riemann member,

$\lambda h/c = 0.370 + 3.047i$, classical fourth-order Runge–Kutta gives $|R(\Delta t \lambda)| > 1$ throughout the scanned range $\Delta t \leq 3 h/c$, so no time step in that range suppresses it. Semi-discrete screening is therefore the appropriate level for this comparison; fully discrete analysis adds time-integration dispersion on top of the spatial results reported here [53], and does not alter the $\Delta t \rightarrow 0$ verdicts. At finite Δt a strongly dissipative integrator could mask the slowest of these rates, in particular the $\mathcal{O}(10^{-3})$ growth of the u - v Riemann/ $\varepsilon = +1$ member, over a finite simulation horizon, and we have not scanned integrators for that possibility.

A two-dimensional spot check reproduces the pattern. The integration-by-parts structure underlying Proposition 2 is dimension-independent, and a preliminary two-dimensional scalar-wave check is consistent with the one-dimensional diagnosis. For the same u - v construction on a homogeneous periodic square cell, detailed together with its validation against the one-dimensional spectra in Appendix B, the tensor-product Q_1 cell is stable to the noise floor, whereas the Q_2 cell grows at a normalized rate of $+7.5$ with the worst mode on the zone diagonal. This reproduces the one-dimensional *pattern*, stability at the lowest order and growth at the next, but not its magnitude, the one-dimensional $p = 2$ rate being $+0.37$. A systematic multidimensional study, including the vector elastodynamic system, remains future work.

The one u - v member that survives to $p = 3$ is first-order dissipative and fails beyond. The combination of Riemann targets with $\varepsilon = +1$, the semi-discrete analog of the spacetime DG pairing of [68], in which both the momentum flux and the kinematic adjoint term carry Riemann values, is the one velocity-trace member free of exponential growth at $p \leq 3$. Its stability does not extend to higher order. Under the arbitrary-precision screening of Appendix C the member grows at normalized rates 1.51×10^{-3} , 1.66×10^{-3} , and 1.66×10^{-3} at $p = 4, 5, 6$, with the $p = 4$ maximum in the interior of the zone at $kh/\pi = 0.725$; the same computation returns 10^{-20} or less at $p \leq 3$. The adjoint term thus postpones the onset of the observed growth and suppresses its rate by three orders of magnitude, but does not remove it, so this member too fails somewhere in the tested range. Even within its stable range the accuracy is limited; as §5 shows, its dissipation error remains at first to barely second order in kh , so it damps resolved waves at the orders where it is stable.

The displacement-trace family follows classical IP theory, with an exact SIPG threshold. With $\eta_I = 1$, NIPG ($\varepsilon = +1$) is stable at all tested orders, while SIPG and IIPG are stable only at $p = 1$; raising the penalty to $\eta_I = 2p(p+1)$ stabilizes both at every order. Rather than infer the coercivity threshold from those two penalty values, we locate it by bisection on η_I at each order. For SIPG it is exactly the triangular number $\eta_I^{\text{crit}} = p(p+1)/2$ at every order from $p = 1$ to 6, so the classical $\mathcal{O}(p^2)$ scaling [21, 85] holds here with an exact rational constant, and the value $\eta_I = 2p(p+1)$ used throughout carries a factor-of-four margin. We report the identification as a measured result at the six orders tested and conjecture that it holds at every order; we have not proved it, and a proof would presumably follow from a sharp inverse estimate for the

trace of the normal derivative on the equispaced nodal basis. For IIPG the threshold is smaller and not rational in p (0, 1.67, 4.22, 7.26, 10.85, 15.04), but fits the same $\mathcal{O}(p^2)$ law with measured exponent 1.99. The alternating-trace variant with $\varepsilon = -1$ behaves like a shifted-stencil SIPG and is likewise stable once penalized. Two caveats apply. First, Riemann targets are dimensionally inconsistent for $w = u$ in the time domain, mixing u with an impedance-scaled velocity, and the resulting members are unstable over most of the range: at every tested order for $\varepsilon = -1$, and from $p = 2$ and $p = 3$ respectively for $\varepsilon = 0$ and $\varepsilon = +1$. They become meaningful only in the frequency domain, and §6 shows that the resulting member remains deficient there. Second, the Baumann–Oden member ($\varepsilon = +1$, $\eta_I = 0$) is stable at $p = 1, 2$ but unstable at $p = 3$ for the wave operator, so stability results for the elliptic operator do not transfer verbatim to elastodynamics.

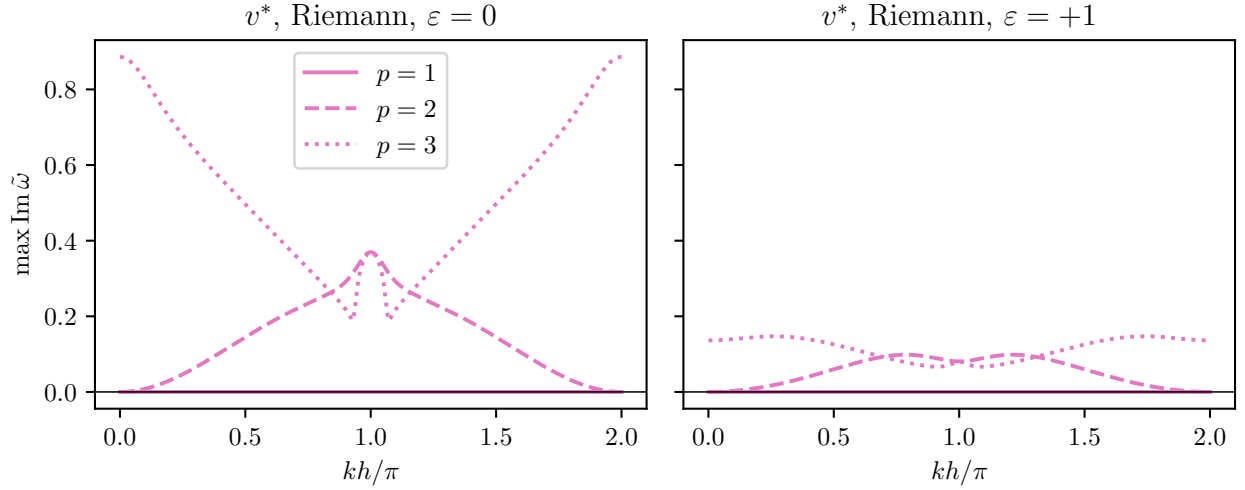


Figure 4: Maximum growth rate $\max_j \operatorname{Im} \omega_j h/c$ over the Brillouin zone for the v^* formulation with Riemann targets, at $\varepsilon = 0$ (left) and $\varepsilon = +1$ (right).

The accuracy comparison of the following sections is therefore restricted to the qualifying set, namely CG (consistent and lumped), the IP family with adequate penalties, the σ – v formulation with all three target families, the u – v Riemann/ $\varepsilon = +1$ member, and, at $p = 1$ only, the v^* members. The qualifying criterion is stability, not merely the absence of a growing Bloch mode, so the two defective members of Table 3 are excluded even at the orders where they carry no exponential growth.

5. Dispersion and dissipation in a homogeneous medium

This section quantifies the accuracy of the qualifying members on a homogeneous medium, where the exact relation is $\omega = ck$ and all errors are functions of kh alone. Figure 5 reports the normalized phase velocity $\operatorname{Re} \omega/(ck)$ and the dissipation $-\operatorname{Im} \omega h/c$ of the physical branch over the resolved range $kh \in (0, \pi]$; the conservative members, namely the $\eta_I = 0$ central and alternating σ – v targets, the IP family, and CG,

do not appear in its right panel at all, and the $u-v$ Riemann member dissipates orders of magnitude more than the $\sigma-v$ Riemann member at every resolved wavenumber. Figure 6 gives the corresponding relative errors (19) in log-log form, whose asymptotic slopes, least-squares fits over $kh \in [0.12, 0.7]$, are collected in Table 4. There a dash marks a conservative member, whose dissipation error sits at the noise floor, and a cross an order at which the member is unstable and therefore not measured.

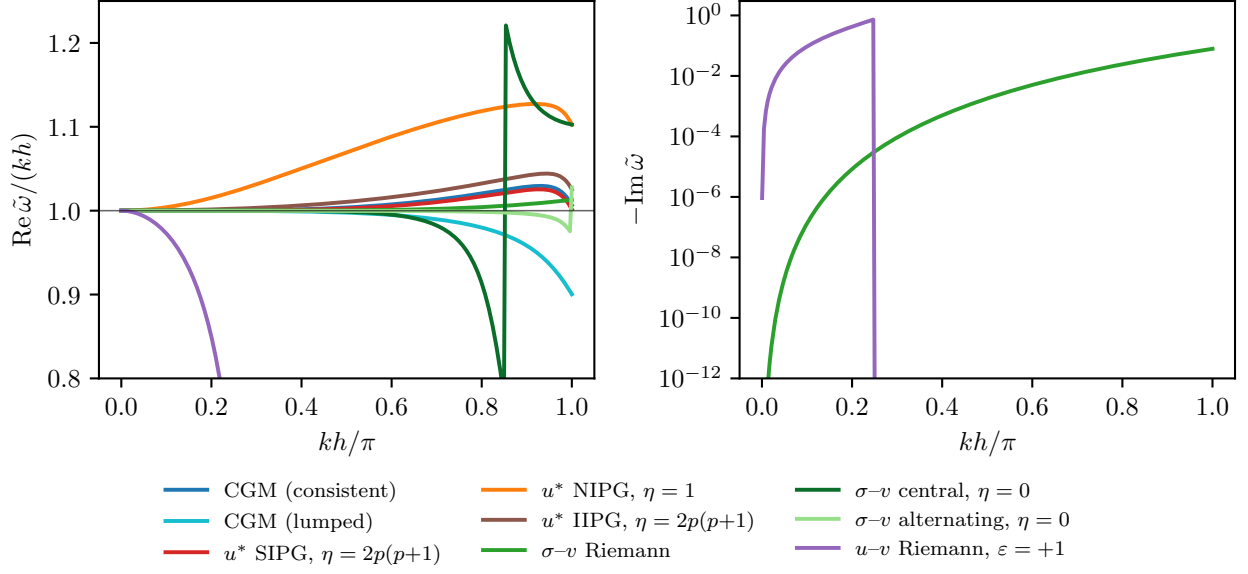


Figure 5: Normalized phase velocity $\text{Re } \omega/(ck)$ (left) and dissipation $-\text{Im } \omega h/c$ (right) of the physical branch in a homogeneous medium at $p = 2$.

Table 4: Measured asymptotic convergence orders of the phase and dissipation errors (19) with respect to kh , in a homogeneous medium.

formulation	$p = 1$		$p = 2$		$p = 3$	
	phase	dissip.	phase	dissip.	phase	dissip.
CG (consistent)	2.0	—	4.0	—	6.0	—
CG (lumped)	2.0	—	4.0	—	3.9	—
u^* SIPG, $\eta = 2p(p+1)$	2.0	—	4.0	—	6.0	—
u^* NIPG, $\eta = 1$	2.0	—	2.0	—	4.0	—
u^* IIPG, $\eta = 2p(p+1)$	2.0	—	2.0	—	4.0	—
$\sigma-v$ Riemann	3.9	3.0	6.0	5.0	8.0	7.0
$\sigma-v$ central, $\eta = 0$	2.0	—	6.0	—	6.0	—
$\sigma-v$ alternating, $\eta = 0$	4.0	—	6.0	—	8.0	—
$u-v$ Riemann, $\varepsilon = +1$	1.9	0.9	2.2	1.1	2.7	1.6
v^* Riemann, $\varepsilon = 0$ ($p=1$)	2.0	1.0	×	×	×	×

The $\sigma-v$ family attains the optimal orders with Riemann and alternating targets. With Riemann targets the phase error converges at order $2p + 2$ and the dissipation error at order $2p + 1$ for every order tested,

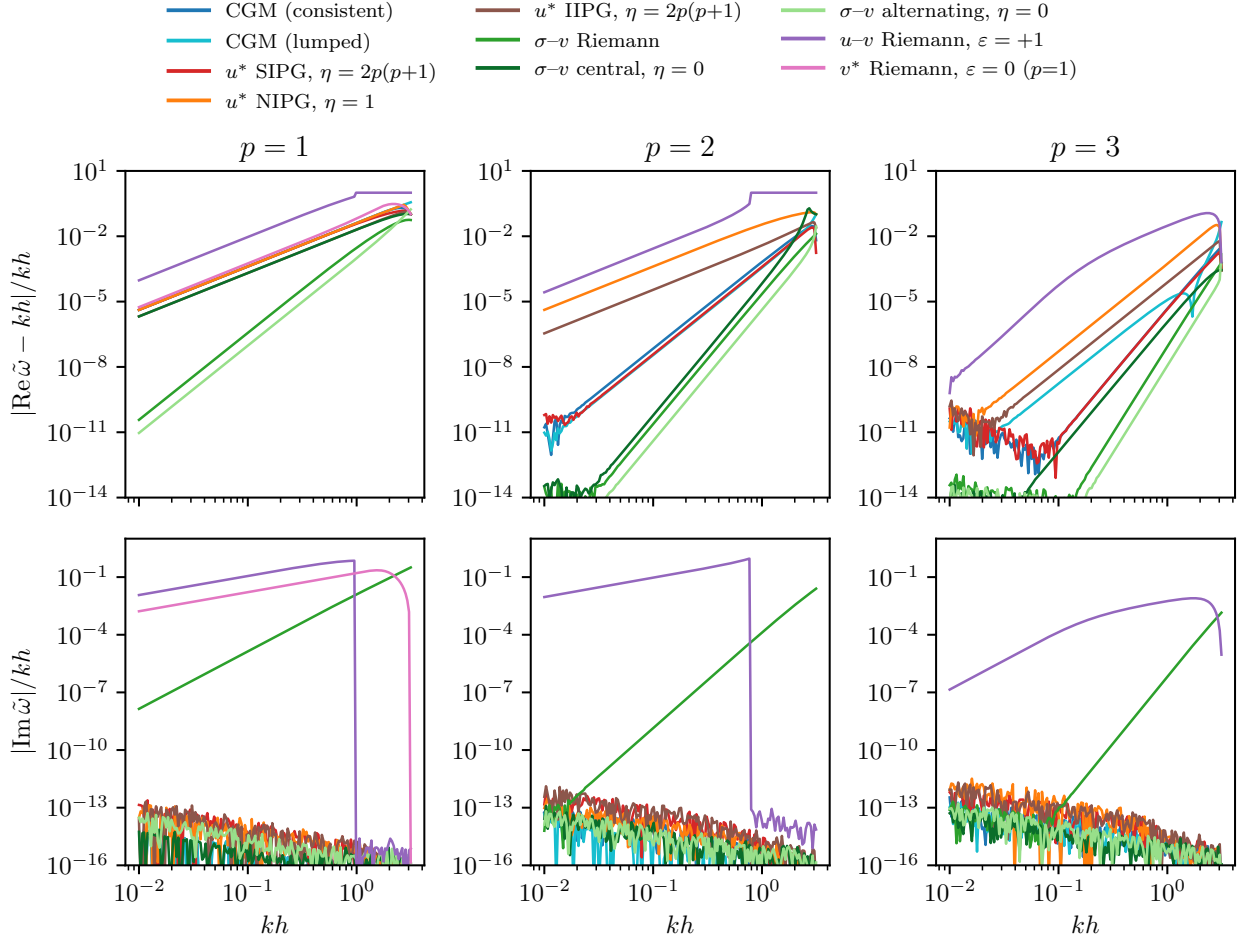


Figure 6: Relative phase error ϵ_{ph} (top) and dissipation error ϵ_{ds} (bottom) of the physical branch versus kh , for $p = 1, 2, 3$ (left to right).

reproducing the classical upwind-DG rates [35, 37, 49]. The alternating targets with $\eta_{\text{I}} = 0$ are exactly conservative and retain the $2p+2$ phase order, the one-dimensional manifestation of LDG superconvergence, whereas central targets exhibit the known parity effect, with order $2p+2$ for even p but $2p$ for odd p .

The displacement-trace family reaches at best the classical $2p$ phase order. SIPG with $\eta_{\text{I}} = 2p(p+1)$ converges at order $2p$, matching CG with consistent mass, whereas NIPG and IIPG reach only order two for $p = 1, 2$ and order four for $p = 3$. The nonsymmetric and incomplete variants lose adjoint consistency and with it show lower observed asymptotic orders, consistent with the grid-dispersion observations of [52] for elastic waves. All IP members are conservative, so their dissipation error vanishes identically; the price is that under-resolved branches are not damped but propagate with $\mathcal{O}(1)$ phase error.

Mass lumping costs two orders at $p = 3$ on equispaced nodes. Lumped CG matches the consistent-mass order $2p$ for $p = 1, 2$, as row-sum lumping of the equispaced quadratic element coincides with Simpson quadrature, but drops to order four at $p = 3$, where the equispaced row-sum weights no longer correspond to an adequate quadrature. Gauss–Lobatto node placement, *i.e.*, the spectral element method [5, 6], is the standard remedy.

The velocity-trace members that qualify are first-order dissipative. The u – v Riemann/ $\varepsilon = +1$ member converges at approximately second order in phase and at first to barely second order in dissipation (measured rates 1.9–2.7 and 0.9–1.6, Table 4). At $kh = 0.1$, about sixty elements per wavelength, it damps the physical branch at a relative rate of 9×10^{-2} per radian at $p = 2$, whereas the σ – v Riemann member of the same degree damps it at 1.4×10^{-9} ; raising the polynomial order barely improves the former, while it improves the latter by four orders per degree. The v^* member at $p = 1$ (its only stable order, with Riemann/ $\varepsilon = 0$ targets) inherits the same defect, with a relative dissipation of 1.7×10^{-2} at $kh = 0.1$, four orders of magnitude more than the σ – v Riemann member of equal order. The velocity-trace construction therefore not only destabilizes most of the family at the tested orders $p = 2, \dots, 6$; the members that remain free of growth damp well-resolved waves at rates that render them unsuitable for wave propagation. A plausible reading, which we have not verified, is that the consistency error of the kinematic statement enters at the interface like a zeroth-order source, so the impedance-weighted velocity jump, which the σ – v form converts into an $\mathcal{O}((kh)^{2p+1})$ dissipation, acts here on an $\mathcal{O}(kh)$ residual.

On a homogeneous medium the σ – v form leads at every degree, the IP family and CG follow at $2p$ with SIPG the best of the interior-penalty variants, mass lumping is free up to $p = 2$ on equispaced nodes and costly beyond, and the velocity-trace members are not competitive even where they are free of growth.

6. A frequency-domain reading of the comparison

Time-harmonic computations read the dispersion relation in the inverse direction. Whereas a transient solver propagates prescribed wavenumber content at the discrete frequencies $\omega_h(k)$, a frequency-domain (Helmholtz) solver imposes a real driving frequency and represents the response through the discrete wavenumbers $k_h(\omega)$ of (21), and the well-known pollution effect of Helmholtz discretizations is precisely the accumulation of the phase error $\text{Re } k_h - \omega/c$ over the propagation distance [46–48]. This section transfers the comparison of §5 to that setting. Beyond the transfer of the accuracy orders, the frequency domain raises two issues that the time-domain results do not settle. The temporal instability of the velocity-trace family has no direct counterpart at real frequency, where no solution component can grow in time, so its manifestation must take a different form. In addition, the Riemann targets for $w = u$, dimensionally inconsistent in the time domain, become meaningful under the time-harmonic substitution $v = -i\omega u$, which

yields the frequency-domain member

$$\sigma^* = \{\sigma\} - \frac{i\omega Z}{2} \llbracket u \rrbracket, \quad u^* = \{u\} + \frac{i}{2Z\omega} \llbracket \sigma \rrbracket, \quad (25)$$

with the interface terms of (7). This is the natural Helmholtz form of the Riemann coupling for a single displacement field, and its relation to the stable σ - v parent and the unstable v^* sibling is established below.

For every stable member the measured frequency-domain accuracy orders, collected in Table 5 as least-squares slopes on the window $\omega h/c \in [0.12, 0.7]$ used for the time-domain fits of Table 4, coincide with those time-domain orders, as the inverse-function relation between $\omega_h(k)$ and $k_h(\omega)$ requires for convergent branches. A dash marks a conservative member, whose attenuation vanishes within its propagating band, and the unstable and defective members are omitted, having no convergent branch to fit; their frequency-domain signature is the anomalous gain discussed below. The σ - v Riemann member converges at order $2p + 2$ in phase and $2p + 1$ in attenuation, the conservative alternating member retains order $2p + 2$ with zero attenuation, SIPG and consistent-mass CG share order $2p$, and NIPG converges at its reduced orders.

Table 5: Measured frequency-domain accuracy orders of the discrete wavenumber $k_h(\omega)$.

formulation	$p = 1$		$p = 2$		$p = 3$	
	phase	atten.	phase	atten.	phase	atten.
CG (consistent)	2.0	–	4.0	–	6.0	–
u^* SIPG	2.0	–	4.0	–	6.0	–
u^* NIPG	2.0	–	1.9	–	4.0	–
σ - v Riemann	4.0	3.0	6.0	5.0	8.0	7.0
σ - v alternating	4.0	–	6.0	–	8.0	–

At ten elements per wavelength and $p = 2$, the relative phase error per wavelength spans 2.5×10^{-7} (σ - v alternating), 1.4×10^{-6} (σ - v Riemann), 6.0×10^{-5} (SIPG), 1.1×10^{-4} (CG), and 1.5×10^{-2} (NIPG), a spread of nearly five orders of magnitude across formulations of identical polynomial order. Since the pollution error accumulates with propagation distance, that spread is the per-wavelength rate at which the formulations separate; converting it into an error on a domain of given length requires a forced finite-domain solve, which we do not perform here.

At real frequency the same members show negative spatial attenuation on selected branches. Figure 7 and Table 6 report the imaginary part of the tracked branch and the worst gain over the resolvable band, restricted to rightward group-velocity branches with the group velocity computed from (22); the restriction excludes the evanescent pairs of band gaps. The gain scan uses 60 frequencies equally spaced over $\omega h/c \in [0.05, 0.98\pi]$, retaining roots subject to the four filters of §3.2, so a zero entry in Table 6 means that no gain was detected on that grid under those filters, not that gain is excluded at every real frequency. In fig. 7 the conservative members lie on the axis within their propagating bands, their positive excursions near a band edge being the evanescent attenuation of a band gap rather than dissipation, and the upwind σ - v member stays nonnegative throughout, whereas the velocity-trace members dip below zero and so amplify

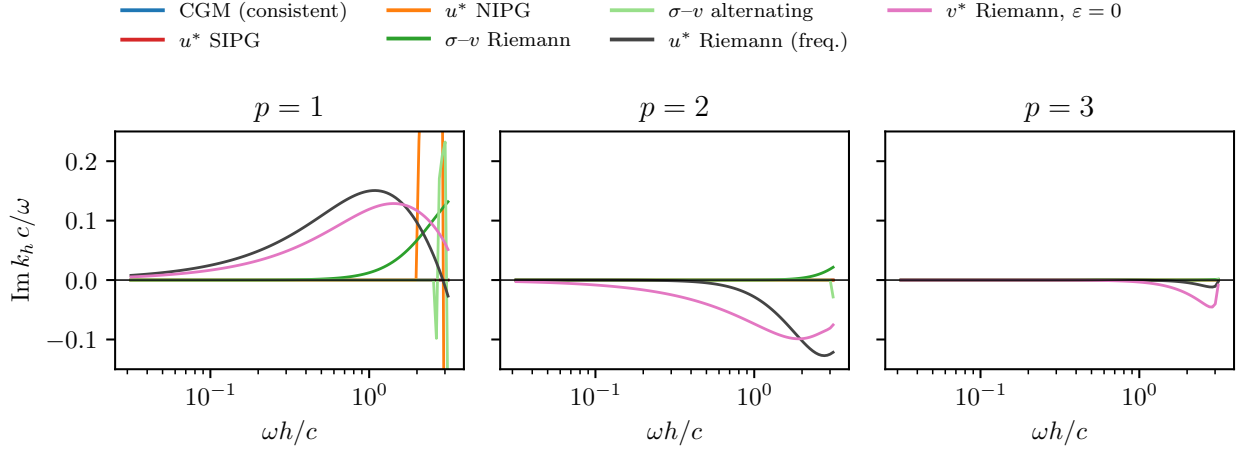


Figure 7: Normalized attenuation $\text{Im } k_h c / \omega$ of the tracked discrete branch at real driving frequency, for $p = 1, 2, 3$ (left to right).

Table 6: Worst anomalous gain $\max(-\text{Im } k_h)$ over the resolvable band, restricted to rightward group-velocity branches, with the frequency at which it occurs.

member	$p = 1$	$p = 2$	$p = 3$
CG (consistent)	0	0	0
u^* SIPG	0	0	0
u^* NIPG	0	0	0
σ - v Riemann	0	0	0
σ - v alternating	0	0	0
u^* Riemann (freq.)	0.06 ($\omega h/c = 3.08$)	0.38 ($\omega h/c = 3.08$)	0.03 ($\omega h/c = 2.92$)
v^* Riemann, $\varepsilon = 0$	0	0.24 ($\omega h/c = 2.98$)	0.13 ($\omega h/c = 2.87$)
u - v Riemann, $\varepsilon = 0$	0	0.24 ($\omega h/c = 2.98$)	0.13 ($\omega h/c = 2.87$)
u - v Riemann, $\varepsilon = +1$	0	0	0

the response along the propagation direction. On the sampled grid the conservative members and the upwind σ - v member register identically zero, and no gain is detected for the lossless NIPG member at any tested order. The u - v and v^* Riemann/ $\varepsilon = 0$ members, by contrast, possess rightward group-velocity branches with $\text{Im } k_h < 0$ at $p = 2$ and $p = 3$, though not at $p = 1$. This parallels the verdicts of §4 qualitatively.⁴

Two numbers, at different frequencies, fix the scale of the effect for these members. At ten elements per wavelength, a standard reference resolution, the $p = 2$ gain is $0.031/h$, an amplification of 3.2% per element. The largest gain anywhere on the sampled grid is $0.243/h$ at $p = 2$, attained at $\omega h/c = 2.98$, and $0.131/h$ at $p = 3$, attained at $\omega h/c = 2.87$, both near the resolution limit; the former corresponds to 27.5% per element. In both cases the selected branch grows rather than decays with distance.⁵

The frequency-domain u^* Riemann member (25) inherits the defect rather than the stability of its parent.

⁴The two are separate computations, and we claim no exact correspondence between them.

⁵Whether a forced solve on a finite domain inherits that behavior is a separate question, which the present periodic-lattice calculation does not settle.

Table 6 reports anomalous gain for this member at every order tested: $0.061/h$ at $p = 1$, $0.379/h$ at $p = 2$, and $0.030/h$ at $p = 3$. The gain is therefore present at every tested order, but not monotone in p . The substitution $v = -i\omega u$ maps the adjoint pairing onto an effective symmetrization weight $\beta = 1/(-i\omega)$, which is again not the energy pairing of Proposition 1, so the defect survives the change of domain, and the conclusion that single-field Riemann coupling requires the energy-paired auxiliary equation, *i.e.*, the σ - v structure, holds in both readings. These results characterize the interior, periodic-lattice behavior that drives pollution. A complete Helmholtz study, with radiation boundary conditions and forced solves, remains outside the present scope.

7. Band structures of layered media

Layered (phononic) media test a formulation against features absent from a homogeneous medium. The unit cell produces band gaps, flat (locally resonant) branches, and strong impedance mismatches at material interfaces, and the exact band structure is available from (24) to machine precision. We use the high-contrast five-layer cell of our companion studies, with layer thicknesses 1.45, 0.5, 0.435, 0.5, 1.45, moduli 8.7, 2.2, 320, 2.2, 8.7, and densities 1.18, 1.1, 8.0, 1.1, 1.18, so that $L = 4.335$ and the wave-speed contrast within the cell reaches 4.5. Element boundaries coincide with material interfaces, each layer is subdivided uniformly, and the Bloch analysis of §3.1 is applied to the full cell.

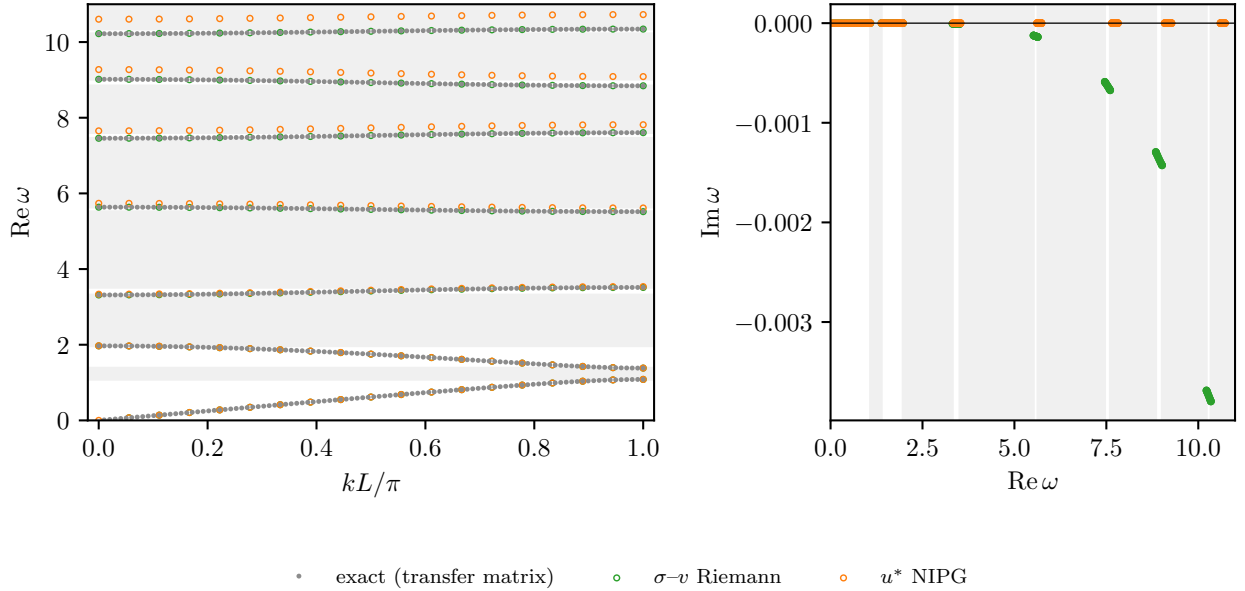


Figure 8: Bloch spectra of the σ - v Riemann and u^* NIPG members (circles) on the exact transfer-matrix band structure (dots), for the high-contrast five-layer cell at $p = 2$ with four elements per layer. Left, $\text{Re } \omega$ with the exact band gaps shaded; right, $\text{Im } \omega$ of the same spectra.

Figure 8 shows the discrete Bloch spectra on the exact band structure at a fixed, moderate resolution.

Both members reproduce the acoustic and lower optical branches and the band-gap edges; the difference lies in the complex plane. The IP member, being conservative, carries every spurious branch on the real axis, indistinguishable from physical ones by frequency alone, whereas the Riemann member pushes them into the lower half plane with damping that grows with band number, which is the spectral signature of upwind dissipation acting on under-resolved content.

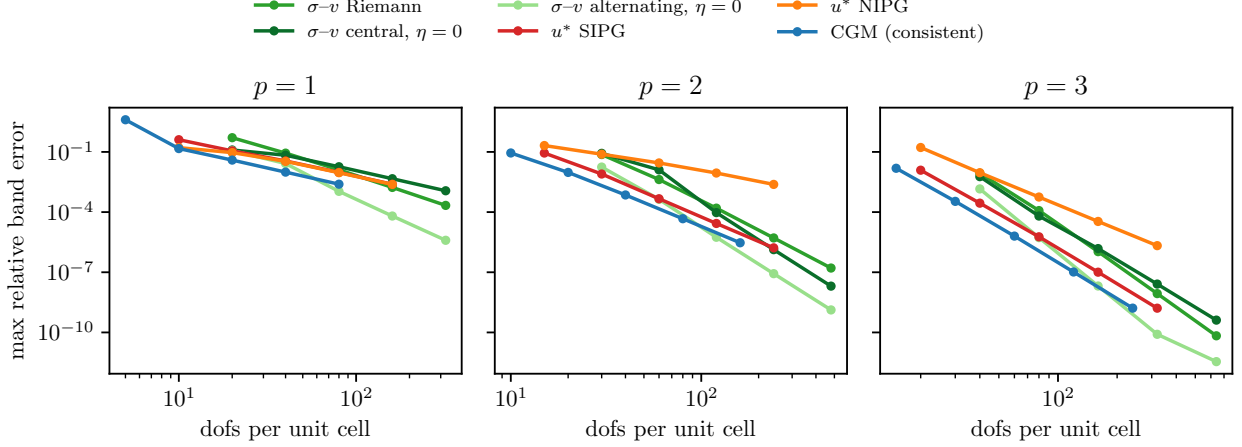


Figure 9: Maximum relative error of the six lowest band frequencies of the layered cell versus degrees of freedom per unit cell, under uniform refinement of the layer subdivision, at $p = 1, 2, 3$ (left to right).

Figure 9 quantifies the maximum error over the six lowest bands under uniform refinement at $p = 1, 2, 3$, measured in the complex plane so that spurious damping of physical bands also registers as error. The metric is the maximum over 18 comparisons, the six lowest exact bands at three wavenumbers of the Brillouin zone, each matched to the discrete spectrum by a one-to-one assignment so that no discrete eigenvalue represents more than one band; the assignment is solved as a linear assignment problem on the relative-error matrix, and any band left unmatched is charged unit relative error. The asymptotic slopes reported below are insensitive to this choice; the coarsest points are not, and no ranking is read from them. The measured convergence orders mirror the homogeneous-medium analysis at every polynomial degree, *including its parity structure*. The conservative σ - v central member attains order $2p + 2$ at even degree but only $2p$ at odd degree: the slopes across the last two refinement levels are 2.00, 6.04, and 5.97 at $p = 1, 2, 3$, reproducing the homogeneous orders 2.0, 6.0, 6.0 of Table 4 to within the resolution of a two-level slope estimate. At $p = 2$ it reaches 2.0×10^{-8} with sixteen elements per layer. The alternating target carries no such parity structure, converging at $2p + 2$ at every degree: the last two levels give 4.02 and 6.01 at $p = 1, 2$, and at $p = 3$ three successive levels give 8.07, 8.02, and 8.00. It is also the most accurate member tested here at every degree; at $p = 3$ with eight elements per layer it reaches 8.1×10^{-11} , against 2.6×10^{-8} for the central member and 8.6×10^{-9} for the Riemann member at the same number of unknowns. Its finest $p = 3$ mesh is the one point in fig. 9 that is not asymptotic: the error there is 3.5×10^{-12} , at which the band metric has

saturated against the precision of the exact reference and of the eigensolver, and the apparent slope falls to 4.52. No order is read from that level.

The other members behave as in the homogeneous case. The σ - v Riemann member converges at order $2p + 1$ at every degree (2.98, 4.98, 6.99). No bimaterial mechanism need be invoked for the loss of one order relative to the central member: the metric is taken in the complex plane, so the $\mathcal{O}((kh)^{2p+1})$ dissipation that Table 4 already records in a homogeneous medium is itself the binding term, and the member is dissipation-limited for the same reason there. SIPG and consistent-mass CG share order $2p$ (1.97, 4.04, 5.96 and 2.00, 3.98, 5.99). NIPG converges at roughly order two for $p = 1, 2$ and order four at $p = 3$ (1.94, 1.89, 4.01), with an error three orders of magnitude above the other members at equal mesh, 8.9×10^{-3} against 5.1×10^{-6} for σ - v Riemann at eight elements per layer. Per degree of freedom the comparison is more nuanced than at equal mesh, since the two-field members carry twice the unknowns and their higher orders tell only with refinement. At the coarsest resolutions the lowest-bands metric is not a meaningful accuracy measure, the discrete spectra being too sparse to represent six bands under the one-to-one assignment; at five degrees of freedom per cell the CG error exceeds unity. What survives refinement is the ordering by asymptotic rate, which is a property of the measured slopes rather than of any one resolution: the σ - v members converge at $2p + 2$ (alternating at every degree, central at even degree) and $2p + 1$ (Riemann), SIPG and CG at $2p$, and NIPG at roughly two. Since the two-field members carry a factor of two in unknowns and a factor of two in unknowns costs at most one refinement level, that ordering is not reversed per degree of freedom once the asymptotic regime is reached, though we have not located the crossover resolution and do not claim a per-degree-of-freedom ranking at coarse mesh. The nonsymmetric member carries the largest error on either measure.

Both plotted members place the band-gap edges accurately well before the optical branches converge pointwise. This is a visual reading of fig. 8; we have not quantified gap-edge error separately, and attach neither a rate nor a mechanism to it.

8. Conclusions

We have placed CG and four discontinuous Galerkin formulations of one-dimensional elastodynamics, namely the displacement-trace (u^*) and velocity-trace (v^*) one-field forms and the displacement-velocity (u - v) and stress-velocity (σ - v) two-field forms, in a single weighted-residual framework in which all interface coupling is expressed through common target-value families (Riemann, central, or alternating, with penalty η_{I} and symmetrization ε), so that each comparison isolates one design choice, and we have compared the members with common Bloch dispersion-dissipation-stability metrics in homogeneous and layered media. The main conclusions are as follows.

First, stability must be screened before accuracy. No sampled member of the velocity-trace family (v^*

and $u-v$) is stable across the tested orders $p = 1, \dots, 6$, though the failure is not at every order for every member and does not always take the same form. Most harbor exponentially growing Bloch modes at every tested order $p = 2, \dots, 6$, and two of the sampled members are unstable already at $p = 1$. The two-field $u-v$ Riemann/ $\varepsilon = +1$ member shows no growth for $p \leq 3$ and grows only at $p = 4, 5, 6$, at rates near $1.6 \times 10^{-3} c/h$, while the $u-v$ central/ $\eta_{\text{I}} = 0/\varepsilon = +1$ member fails in the opposite direction, exponentially unstable at $p = 1, 2, 3$ and merely defective at $p = 4, 5, 6$. Growth therefore strengthens monotonically with polynomial order only along the Riemann/ $\varepsilon = 0$ and central/ $\varepsilon \in \{-1, 0\}$ sequences, not across the family. Two members fail in a weaker but still disqualifying way, carrying no growing Bloch mode but a defective $\omega = 0$ eigenvalue and hence algebraic growth in time; for the central/ $\eta_{\text{I}} = 0/\varepsilon = 0$ member, which is exponentially unstable at no tested order, the defect worsens with polynomial degree, so refining in order makes the member worse rather than better. A scan over a 17×21 grid of the penalty-symmetrization plane for the symmetrized v^* member shows growth at all 356 non-degenerate sampled pairs; this is a finite grid scan, not a continuous exclusion. The instability is an interior property, invisible to frequency-only spectrum plots, and for the unsymmetrized members we trace it to the kinematic statement, whose mass pairing of $\dot{u} - v$ forfeits the compatibility flux that the energy estimate of the first-order system requires; for $\varepsilon \neq 0$ the observed behavior is consistent with that mechanism rather than derived from it. Read as a design requirement rather than as a verdict on particular members, this says that the auxiliary equation of a two-field formulation must be tested in the pairing that makes it conjugate to the momentum equation in the energy; consistency of the kinematic statement alone is not sufficient. The displacement-trace family follows classical interior-penalty theory, with a measured SIPG coercivity threshold of exactly $p(p+1)/2$ and an IIPG threshold of the same $\mathcal{O}(p^2)$ order, while the Baumann-Oden member is unstable at $p = 3$ for the wave operator.

Second, among the stable members the energy-paired $\sigma-v$ formulation offers the most favorable combination of stability margin, spectral accuracy, and control of nonphysical branches, although it carries twice the unknowns of the one-field members, so that its advantage per degree of freedom tells only once the spectrum is resolved. On phase accuracy alone the alternating targets are the better of the two $\sigma-v$ variants, matching or exceeding Riemann at every order in homogeneous, frequency-domain, and layered settings alike, and giving the smallest frequency-domain phase error at $p = 2$; the case for Riemann targets rests instead on their damping of under-resolved and spurious content, which the conservative variants leave undamped on the real axis. In homogeneous media Riemann targets deliver phase and dissipation accuracy of orders $2p+2$ and $2p+1$, and alternating targets conserve energy exactly while retaining order $2p+2$ at every degree. On the layered cell the Riemann member becomes dissipation-limited at $2p+1$ through the material interfaces, the central member reproduces the homogeneous parity pattern ($2p+2$ at even degree, $2p$ at odd), the alternating member retains $2p+2$ at every degree and is the most accurate member tested, and the nonsymmetric IP member carries errors orders of magnitude larger than the other members at equal

mesh. The one u - v member free of growth at low order, in contrast, dissipates resolved waves at first order and is therefore unsuitable for wave propagation even where it is stable.

Third, the comparison transfers to the frequency domain through the inverse Bloch eigenproblem. The accuracy orders of the stable members carry over to the discrete wavenumber, with phase errors at ten elements per wavelength spanning five orders of magnitude across formulations of equal polynomial order, and at real frequency the velocity-trace family shows negative spatial attenuation on rightward group-velocity branches; the frequency-domain Riemann coupling of a single displacement field, the natural Helmholtz form of the u^* Riemann member, inherits the defect already at $p = 1$. The sampled inverse-Bloch spectra thus exhibit anomalous spatial gain on selected rightward group-velocity branches, qualitatively paralleling the temporal instability results.

The comparison supports the following recommendations. When a two-field discretization is acceptable, the stress-velocity form is the best performing of the methods, orders, and benchmarks tested: Riemann targets where spurious branches must be damped, alternating targets where phase accuracy matters most and energy conservation is wanted. When a single displacement field is required, SIPG with an adequate penalty matches consistent-mass CG at equal mesh, though DG carries $p + 1$ unknowns per element against CG's p , and NIPG/IIPG show lower observed asymptotic orders. The mass-paired displacement-velocity family analyzed here should be avoided unless its kinematic statement is recast in the energy pairing of the stress-velocity form: two of its members are already unstable at $p = 1$, and none of the sampled members is stable across the tested range.

The scope of these conclusions should be read narrowly. Everything reported here is one-dimensional, scalar, and semi-discrete, on uniform meshes whose element boundaries coincide with material interfaces. The analysis is interior: it uses a periodic cell, imposes no boundary conditions, and performs no forced solve, so it characterizes the pollution-driving behavior of the interior stencil rather than the response of a bounded domain. Every stability verdict is a finite scan, over a finite wavenumber grid and a finite set of parameters, and is therefore an exclusion only over what was sampled. The layered study uses a single unit cell. Extensions of the present unified comparison to two dimensions and the full tensor system, to the fully discrete setting, to randomly heterogeneous media, in which formulation bias must be weighed against realization spread and which requires random-field ensembles with verified pointwise statistics, and to the spacetime discontinuous Galerkin methods from which the Riemann target values originate are natural next steps.

Appendix A. Parent element matrices

All formulations use nodal Lagrange elements on the parent domain $\xi \in [-1, 1]$ with the degree-of-freedom ordering [left end, right end, interior nodes left to right]; the interior nodes are equispaced. With

$\mathbf{m} = \int_{-1}^1 \mathbf{N} \mathbf{N}^T d\xi$, $\mathbf{k} = \int_{-1}^1 \mathbf{N}' \mathbf{N}'^T d\xi$, and the mixed gradient-value matrix $\mathbf{g} = \int_{-1}^1 \mathbf{N}' \mathbf{N}^T d\xi$ required by the σ - v formulation (11), the element matrices of an element of size h_e follow as $(h_e/2) \mathbf{m}$, $(2/h_e) \mathbf{k}$, and $\mathbf{g}$ (the Jacobians cancel in the mixed matrix). For $p = 1$:

$$\mathbf{m} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{k} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{g} = \frac{1}{2} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}; \quad (\text{A.1})$$

for $p = 2$ (ordering [L, R, mid]):

$$\mathbf{m} = \frac{1}{15} \begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix}, \quad \mathbf{k} = \frac{1}{6} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}, \quad \mathbf{g} = \frac{1}{6} \begin{bmatrix} -3 & 1 & -4 \\ -1 & 3 & 4 \\ 4 & -4 & 0 \end{bmatrix}; \quad (\text{A.2})$$

and for $p = 3$ (ordering [L, R, $\xi = -\frac{1}{3}$, $\xi = +\frac{1}{3}$]):

$$\begin{aligned} \mathbf{m} &= \frac{1}{840} \begin{bmatrix} 128 & 19 & 99 & -36 \\ 19 & 128 & -36 & 99 \\ 99 & -36 & 648 & -81 \\ -36 & 99 & -81 & 648 \end{bmatrix}, & \mathbf{k} &= \frac{1}{80} \begin{bmatrix} 148 & -13 & -189 & 54 \\ -13 & 148 & 54 & -189 \\ -189 & 54 & 432 & -297 \\ 54 & -189 & -297 & 432 \end{bmatrix}, \\ \mathbf{g} &= \frac{1}{80} \begin{bmatrix} -40 & -7 & -57 & 24 \\ 7 & 40 & -24 & 57 \\ 57 & 24 & 0 & -81 \\ -24 & -57 & 81 & 0 \end{bmatrix}. \end{aligned} \quad (\text{A.3})$$

The identity $\mathbf{g} + \mathbf{g}^T = \mathbf{N}(1)\mathbf{N}(1)^T - \mathbf{N}(-1)\mathbf{N}(-1)^T$ (integration by parts) provides a check of these values. Row-sum lumping replaces $\mathbf{m}$ by $\text{diag}(1, 1)$, $\frac{1}{3} \text{diag}(1, 1, 4)$, and $\frac{1}{4} \text{diag}(1, 1, 3, 3)$ for $p = 1, 2, 3$.

Appendix B. Two-dimensional spot check of the velocity-trace instability

This appendix details the preliminary two-dimensional check referenced in §4. The model problem is the scalar (anti-plane shear) wave equation $\rho \dot{v} = \nabla \cdot (E \nabla u)$ on a homogeneous periodic square cell of side h with $E = \rho = 1$. The discretization mirrors the one-dimensional u - v member. Displacement and velocity are interpolated by tensor-product Q_1 or Q_2 Lagrange elements; the kinematic relation $\dot{u} = v$ is enforced with the mass pairing scaled by $\alpha = (c/h)^2$ and carries no interface term ($\varepsilon = 0$); and the momentum equation is tested by $\hat{v}$ with the Riemann momentum flux

$$\sigma_n^* = \{E \nabla u \cdot \mathbf{n}\} + \frac{Z}{2} \llbracket v \rrbracket \quad (\text{B.1})$$

on each of the four edges, in which $\mathbf{n}$ is the fixed edge normal, the average and jump are taken across the edge in the orientation of $\mathbf{n}$, and the edge integrals are evaluated by Gauss quadrature. Bloch periodicity

multiplies the neighbor traces by $e^{ik_x h}$ and $e^{ik_y h}$ on the right and top edges and by the inverse factors on the left and bottom edges, and the semi-discrete cell problem $\mathbf{M}\dot{\mathbf{a}} + \mathbf{K}(\mathbf{k})\mathbf{a} = \mathbf{0}$ is assembled from the same parent matrices as in [Appendix A](#) by tensor products.

The implementation is validated in three steps. The edge assembly by quadrature agrees with an independent closed-form Kronecker-product assembly to 10^{-15} ; at $k_y = 0$ the two-dimensional spectrum contains the one-dimensional u - v branches of §4, with the physical frequencies matching to 10^{-12} and the maximal growth rates agreeing at every sampled k_x ; and the Q_1 cell is stable over 400 random points of the two-dimensional Brillouin zone (maximal growth at the eigensolver noise floor). With the machinery so validated, scanning $(k_x, k_y) \in [0, 2\pi/h)^2$ on grids up to 41×41 yields a maximal normalized growth rate of +7.5 for the Q_2 cell, attained on the zone diagonal $k_x = k_y$ and converging from +6.7 on coarser scans, whereas the Q_1 cell remains stable. This reproduces the *qualitative* one-dimensional pattern, stability at the lowest order and exponential growth at the next, for the mass-paired u - v formulation in two dimensions. The rates themselves are not comparable: +7.5 here against +0.37 for the one-dimensional $p = 2$ member, a factor of twenty, so we draw no quantitative inference from a single cell geometry and element family.

Appendix C. Higher-order stability scans

Table C.7 extends the stability screening of §4 to $p = 4, 5, 6$ for the velocity-trace members, using the single-cell analysis of §3.4 and the same three outcome symbols as Table 2. The scan covers the v^* and u - v formulations with Riemann and central targets at $\varepsilon \in \{-1, 0, +1\}$ and $\eta_{\text{I}} \in \{0, 1\}$, together with both alternating orientations at $\varepsilon = 0$, $\eta_{\text{I}} = 0$; the remaining alternating symmetrizations were not scanned at high order.

At these orders the magnitude of a computed rate carries no information about whether it is real. A Jordan block of size m lifts the computed spectrum by $\epsilon^{1/m}$ for machine epsilon ϵ : that is 10^{-8} at $m = 2$, but 2.2×10^{-3} at $m = 6$ and exactly 10^{-2} at $m = 8$, and blocks of size eight occur here (Table 3). We therefore build the parent element matrices in exact rational arithmetic and diagonalize the Bloch pencil at two working precisions, 18 and 36 decimal digits, classifying by *precision invariance*: a genuine eigenvalue does not move when the precision doubles, whereas roundoff collapses.

The procedure is staged, the stages differing in cost. A double-precision sweep over 161 equally spaced wavenumbers locates the maximum of $\text{Im} \omega h/c$ first, and the pencil is re-diagonalized at both working precisions *at that maximum*. If the two agree to six significant digits the sweep value stands. If they do not, the pencil is re-diagonalized at both precisions on a reduced grid of 25 uniform wavenumbers together with seven points refining the neighborhood of the double-precision maximum, and the member is reported unstable when the two precisions agree to six significant digits, stable when the higher precision falls below 10^{-18} , and defective otherwise. Both raw values are archived, and a size-two degeneracy at the zone center

is not reported as a defect, being the rigid-body pair of §3.5 rather than a property of the discretization. Applied to $p = 1, 2, 3$ the same code path generates every entry of Table 2.

Three conclusions follow, and all three revise the picture that a double-precision scan suggests. First, the two-field u - v Riemann/ $\varepsilon = +1$ member, the only velocity-trace member free of exponential growth at $p \leq 3$, is *not* an exception at higher order. Its growth rate at $p = 4, 5, 6$ is 1.51×10^{-3} , 1.66×10^{-3} , and 1.66×10^{-3} . These values are large enough to be visible in double precision; what the high-precision computation establishes is that they are *genuine* rather than the roundoff signature of a degeneracy, which double precision alone cannot decide. They are unchanged to the twelve digits reported when the working precision is doubled, and reproduced to a relative 10^{-25} by a modal (Legendre) assembly spanning the same discrete space. At $p = 4$ the maximum is attained at $kh/\pi = 0.725$, in the interior of the Brillouin zone, and the growth is positive over almost the whole zone rather than confined to its endpoints. At $p \leq 3$ the same computation returns 10^{-20} or less, the residue of the physical rigid-body pair at $k = 0$. The adjoint term therefore postpones the onset of the observed growth from $p = 2$ to $p = 4$ and suppresses its rate by three orders of magnitude, but does not remove it.

Equation (10) fixes the kinematic weight at $\alpha_e = (c_e/h_e)^2$, but that choice sets the relative weight of the adjoint term whenever $\varepsilon \neq 0$, so we test its influence by introducing a dimensionless multiplier

$$\alpha_e = \chi (c_e/h_e)^2, \tag{C.1}$$

with $\chi = 1$ the nominal value, and scanning $\chi \in \{0.01, 0.1, 0.25, 0.5, 1, 2, 4, 10, 100\}$. At $\varepsilon = 0$ the scan confirms that χ leaves the spectrum invariant: the maximal growth is 0.3697029378 at $p = 2$ and 1.564204664 at $p = 4$ for $\chi \in \{0.1, 1, 10\}$, identical to ten digits. At $\varepsilon = +1$ the growth is instead *proportional* to χ over the moderate range, measuring 1.51×10^{-5} , 1.51×10^{-4} , 1.51×10^{-3} , and 2.99×10^{-3} at $p = 4$ for $\chi = 0.01, 0.1, 1, 2$, and growing sublinearly beyond. A small χ therefore suppresses the growth in proportion but does not remove it at any sampled value, and the linear trend gives no indication of a root at positive χ ; nine values over four decades are not an exclusion for every $\chi > 0$. A large χ makes matters worse rather than better: at $\chi = 100$ the $p = 3$ member, stable at the nominal $\chi = 1$, also acquires exponential growth. This is consistent with the missing-flux mechanism of Proposition 2, in which χ weights a mass-inverse lifting and not the absent $\llbracket \sigma \rrbracket$ flux.

Second, the converse also fails: the u - v central/ $\varepsilon = +1$, $\eta_{\text{I}} = 0$ member is exponentially unstable at $p = 1, 2, 3$ with rates of order unity, yet carries no exponential growth at $p = 4, 5, 6$, where instead its $\omega = 0$ eigenvalue is defective in the sense just described, giving growth linear in time rather than exponential. Growth rates are therefore not monotone in the polynomial order, and monotone strengthening should be asserted only for the Riemann/ $\varepsilon = 0$ and central/ $\varepsilon \in \{-1, 0\}$ sequences in which it is observed.

Third, the central/ $\eta_{\text{I}} = 0/\varepsilon = 0$ member is exponentially unstable at no tested order, failing instead through the growing defect of §4. In double precision its rates read 2.1×10^{-2} , 4.8×10^{-2} , and 1.5×10^{-1}

at $p = 4, 5, 6$. None of the three survives a change of working precision: all are roundoff at a block of size six to eight, and all are large enough to pass a test on magnitude alone.

No sampled member of the velocity-trace family is stable across $p = 1, \dots, 6$. All but two fail by exponential growth at one or more orders; the remaining two fail by a defective $\omega = 0$ eigenvalue, which is a weaker failure than exponential growth but is still a failure, since it produces unbounded algebraic growth of the semi-discrete solution.

Table C.7: Maximal normalized Bloch growth rate $\max_k \text{Im } \omega h/c$ of the velocity-trace members at $p = 4, 5, 6$, on a homogeneous single cell.

formulation	target family	ε	$p = 4$	$p = 5$	$p = 6$
v^*	Riemann	-1	3.90×10^2	7.98×10^2	1.46×10^3
v^*	central, $\eta_I = 0$	-1	4.20×10^2	8.40×10^2	1.51×10^3
v^*	central, $\eta_I = 1$	-1	3.90×10^2	7.98×10^2	1.46×10^3
v^*	Riemann	+0	1.56×10^0	2.46×10^0	3.58×10^0
v^*	central, $\eta_I = 0$	+0	$\dagger_6$	$\dagger_6$	$\dagger_8$
v^*	central, $\eta_I = 1$	+0	1.56×10^0	2.46×10^0	3.58×10^0
v^*	Riemann	+1	1.77×10^{-1}	1.98×10^{-1}	2.14×10^{-1}
v^*	central, $\eta_I = 0$	+1	1.86×10^{-1}	2.07×10^{-1}	2.21×10^{-1}
v^*	central, $\eta_I = 1$	+1	1.71×10^{-1}	1.94×10^{-1}	2.11×10^{-1}
v^*	alternating (σ^L)	+0	3.33×10^0	3.96×10^0	4.59×10^0
v^*	alternating (σ^R)	+0	3.33×10^0	3.96×10^0	4.59×10^0
$u-v$	Riemann	-1	7.56×10^3	2.18×10^4	5.34×10^4
$u-v$	central, $\eta_I = 0$	-1	2.56×10^2	5.36×10^2	9.91×10^2
$u-v$	central, $\eta_I = 1$	-1	2.41×10^2	5.15×10^2	9.64×10^2
$u-v$	Riemann	+0	1.56×10^0	2.46×10^0	3.58×10^0
$u-v$	central, $\eta_I = 0$	+0	$\dagger_6$	$\dagger_6$	$\dagger_8$
$u-v$	central, $\eta_I = 1$	+0	1.56×10^0	2.46×10^0	3.58×10^0
$u-v$	Riemann	+1	1.51×10^{-3}	1.66×10^{-3}	1.66×10^{-3}
$u-v$	central, $\eta_I = 0$	+1	$\dagger_2$	$\dagger_2$	$\dagger_2$
$u-v$	central, $\eta_I = 1$	+1	5.85×10^{-2}	8.43×10^{-3}	2.32×10^{-3}
$u-v$	alternating (σ^L)	+0	3.33×10^0	3.96×10^0	4.59×10^0
$u-v$	alternating (σ^R)	+0	3.33×10^0	3.96×10^0	4.59×10^0

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The 1D solver, the arbitrary-precision stability and Jordan-structure scripts, the analysis scripts, and the cached data reproducing every figure and table in this paper will be deposited in a public repository with a citable DOI on acceptance, and are available from the corresponding author in the interim.

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