Istanbul Technical University

Graduate School of Science Engineering and Technology

MKC525E
Foundations of Solid Mechanics

Final Exam

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1 Question 1

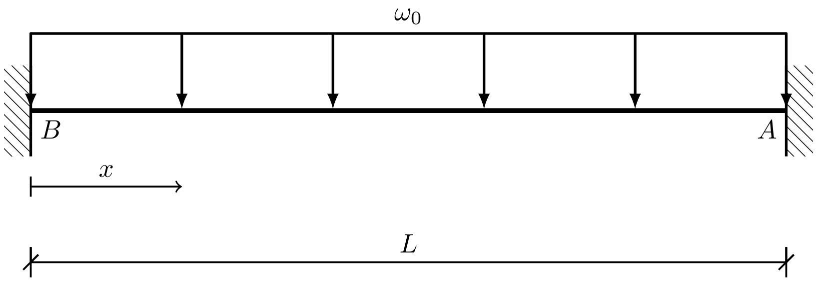

Figure 1: Uniform flexible beam clamped at both ends and loaded with a uniformly distributed load ω_0.

A beam clamped at the both ends subjected to an uniformly distributed load ω_0 over its entire span is shown in Figure 1.
The bending moment $M_x(x, x)$ should be determined in order to solve the deflection of the beam. To do this, we should look at the free body diagrams and find the reaction forces and moments.

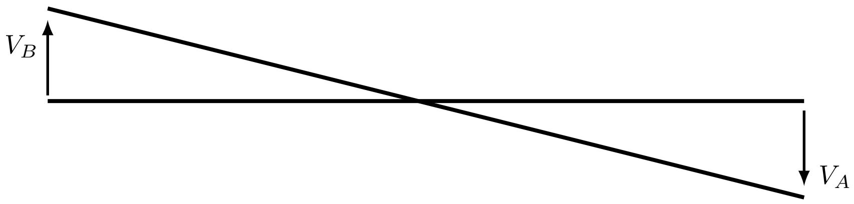

Figure 2: Shear force diagram.

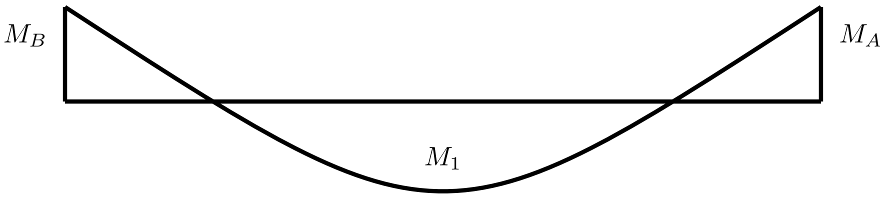

Figure 3: Bending moment diagram.

The shear force diagram and the bending moment diagram of the beam is given in Figures 2 and 3 respectively.
Reaction forces:

$$V_B = \frac{\omega_0 L}{2} \qquad \text{and} \qquad V_A = \frac{\omega_0 L}{2} \tag{1.1}$$

Reaction moments and bending moment at the centre:

$$M_B = -\frac{\omega_0 L^2}{12} \qquad \text{and} \qquad M_1 = \frac{\omega_0 L^2}{24} \qquad \text{and} \qquad M_A = -\frac{\omega_0 L^2}{12} \tag{1.2}$$

The bending moment M_x at any distance $0 \leq x \leq L$ where L is the length of the beam as follows:

$$M_x = M_B + M_{\omega_0} + V_B \tag{1.3}$$

$$= -\frac{\omega_0 L^2}{12} - \frac{\omega_0 x^2}{2} + \frac{\omega_0 L x}{2} \tag{1.4}$$

$$= \frac{\omega_0}{12}\left\{6Lx - L^2 - Lx^2\right\} \tag{1.5}$$

It should be noted that there is no horizontal movement in this particular problem. So:

$$\Delta = 0 \tag{1.6}$$
$$x_0 = x + \Delta \qquad \Rightarrow \qquad x_0 = x \tag{1.7}$$

Non-linear Euler-Bernoulli differential equation:

$$\frac{y'' EI}{[1+(y')^2]^{3/2}} = -\frac{M}{EI} \tag{1.8}$$

By substituting Eq. 1.5 into Eq. 1.8:

$$\frac{y'' EI}{\left[1+(y')^2\right]^{3/2}} = \frac{\omega_0}{12}\left\{6Lx - L^2 - Lx^2\right\} \tag{1.9}$$

If $y' \ll 1$:

$$y'' EI = \frac{\omega_0}{12}\left\{6Lx - L^2 - Lx^2\right\} \tag{1.10}$$
$$y' EI = \frac{\omega_0}{12}\left\{3Lx^2 - L^2 x - \frac{Lx^3}{3}\right\} + C_1 \tag{1.11}$$
$$yEI = \frac{\omega_0}{12}\left\{Lx^3 - L^2x^2/2 - \frac{Lx^4}{12}\right\} + C_1 + C_2 \tag{1.12}$$
$$= \frac{\omega_0 x^2}{24}(L-x)^2 \tag{1.13}$$

$$y|_{x=L} = 0 \implies C_1 + C_2 = 0 \tag{1.14}$$
$$y'|_{x=L} = 0 \implies C_1 = 0 \tag{1.15}$$

Otherwise:

$$\frac{y'' EI}{\left[1+(y')^2\right]^{3/2}} = \frac{\omega_0}{12}\left\{6Lx - L^2 - Lx^2\right\} \tag{1.16}$$

Let's $y' = p$ and $y'' = p'$:

$$\frac{p' EI}{\left[1+(p)^2\right]^{3/2}} = \frac{\omega_0}{12}\left\{6Lx - L^2 - Lx^2\right\} \int \frac{\mathbf{d}p EI}{\left[1+(p)^2\right]^{3/2}} = \int \frac{\omega_0}{12}\left\{6Lx - L^2 - Lx^2\right\} \tag{1.17}$$

Let's $p = \tan\theta$ and $\mathbf{d}p = \sec^2\theta \mathbf{d}\theta$:

$$\int \frac{\sec^2\theta \mathbf{d}\theta}{\left[1+\frac{\sin^2\theta}{\cos^2\theta}\right]^{3/2}} = \frac{\omega_0}{12}\left\{6Lx - L^2 - Lx^2\right\} \tag{1.18}$$

$$\int \cos\theta \mathbf{d}\theta = \frac{\omega_0}{12}\left\{6Lx - L^2 - Lx^2\right\} \tag{1.19}$$

$$\sin\theta = \frac{\omega_0}{12}\left\{3Lx^2 - L^2x - \frac{Lx^3}{3}\right\} + C_1 \tag{1.20}$$

$$\frac{p'EI}{[1+(p)^2]^{3/2}} = \frac{\omega_0}{12}\left\{3Lx^2 - L^2x - \frac{Lx^3}{3}\right\} + C_1 \tag{1.21}$$

$$\frac{y''EI}{[1+(y')^2]^{3/2}} = \frac{\omega_0}{12}\left\{3Lx^2 - L^2x - \frac{Lx^3}{3}\right\} + C_1 \tag{1.22}$$

$$y''|_{x=0} = 0 \implies \tag{1.23}$$

$$y'|_{x=0} = 0 \implies C_1 = 0 \tag{1.24}$$

$$y' = \frac{\frac{\omega_0}{12}(3Lx^2 - L^2x - \frac{Lx^3}{3}) + EI}{\sqrt{1-(\frac{\omega_0}{12}3Lx^2 - L^2x - \frac{Lx^3}{3} + EI)^2}} \tag{1.25}$$

$$y = \int_0^x \frac{\frac{\omega_0}{12}\left\{3Lx^2 - L^2x - \frac{Lx^3}{3}\right\} + EI}{\sqrt{1-\left\{\frac{\omega_0}{12}(3Lx^2 - L^2x - \frac{Lx^3}{3}) + EI\right\}^2}}\mathbf{d}x \tag{1.26}$$

Analytical solution:

$$\omega_0|_{x=\frac{L}{2}} = \frac{\omega_0 L^4}{384EI} \tag{1.27}$$

Using nonlinear theory[1]:

$$\mathrm{N(x) = N_A \cos\theta + Q_A \sin\theta - \omega_0 \sin\theta \int_0^x dx} \tag{1.28}$$

$$\mathrm{Q(x) = N_A \sin\theta - Q}_A \sin\theta + \omega_0 \cos\theta \int_0^x \mathrm{d}x \tag{1.29}$$

$$\mathrm{M(x) = -Q_A x + N_A v + \frac{\omega_0 x^2}{2} + M_A} \tag{1.30}$$

where N is the normal force, Q is the shear force and M is the bending moment.
At the middle of the beam, $\theta = 0$

$$\mathrm{N(0) = N_A = N_B} \tag{1.31}$$

[1]See Fertis

2 Question 2

The compatibility equations (Saint-Venant's compatibility equations) for the engineering strain ε_{ij} is given by:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{kj,il} - \varepsilon_{il,kj} = 0; \qquad (i,j,k,l) = 1,2,3 \tag{2.1}$$

In expanded form:

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{kj}}{\partial x_i \partial x_l} - \frac{\partial^2 \varepsilon_{il}}{\partial x_k \partial x_j} = 0 \tag{2.2}$$

Non-trivial relations:

$$\begin{array}{cccccc} i & \neq & j & \text{and} & k & \neq & l \\ 1 & & 2 & \text{and} & 1 & & 2 \\ 2 & & 3 & \text{and} & 2 & & 3 \\ 3 & & 1 & \text{and} & 2 & & 1 \\ 1 & & 2 & \text{and} & 1 & & 3 \\ 2 & & 3 & \text{and} & 2 & & 1 \\ 3 & & 1 & \text{and} & 3 & & 2 \end{array} \tag{2.3}$$

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \tag{2.4}$$
$$\varepsilon_{22,33} + \varepsilon_{33,22} = 2\varepsilon_{23,23} \tag{2.5}$$
$$\varepsilon_{33,11} + \varepsilon_{11,33} = 2\varepsilon_{31,31} \tag{2.6}$$
$$\varepsilon_{12,23} + \varepsilon_{23,12} = \varepsilon_{22,31} + \varepsilon_{31,22} \tag{2.7}$$
$$\varepsilon_{23,31} + \varepsilon_{31,23} = \varepsilon_{33,12} + \varepsilon_{12,33} \tag{2.8}$$
$$\varepsilon_{31,12} + \varepsilon_{12,31} = \varepsilon_{11,23} + \varepsilon_{23,11} \tag{2.9}$$

Equations of equilibrium:

$$\sigma_{ij,j} + p_i = 0 \tag{2.10}$$

Constitutive Law:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2.11}$$
$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \tag{2.12}$$

If the material is linear isotropic:

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{2.13}$$
$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] \tag{2.14}$$

By substituting Eq. 2.11 into Eq. 2.1:

$$((S_{ijmn}\sigma_m))_{kl} + (S_{klmn}\sigma_{mn})_{ij} = (S_{ikmn}\sigma_{mn})_{jl} + (S_{ilmn}\sigma_{mn})_{ik} \tag{2.15}$$

Plane stress strain field implies that stress is constant through thickness ($\frac{\partial}{\partial_{x_3}} = 0$) and;

$$\sigma_{31} = \sigma_{13},$$
$$\sigma_{32} = \sigma_{23}$$
$$\sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \tag{2.16}$$
$$\varepsilon_{31} = \varepsilon_{32} = 0 \tag{2.17}$$

Eq. 2.10 reduces to:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + p_1(x_1, x_2) = 0 \tag{2.18}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + p_2(x_1, x_2) = 0 \tag{2.19}$$

Compatibility conditions for small strain plane stress:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 0 \tag{2.20}$$

In index notation:

$$e_{ik,jl} + e_{jl,ik} - e_{jk,il} - e_{il,jk} = 0 \tag{2.21}$$

$$\begin{array}{ccccccc} i & \neq & j & \text{and} & k & \neq & l \\ 1 & & 2 & \text{and} & 1 & & 2 \end{array} \tag{2.22}$$

Compatibility conditions for large strain:

$$R^{(\mathrm{C})M}_{IJK} = \Gamma^{(\mathrm{C})M}_{IK,J} - \Gamma^{(\mathrm{C})M}_{IJ,K} + \Gamma^{(\mathrm{C})M}_{LJ}\Gamma^{(\mathrm{C})L}_{IK} - \Gamma^{(\mathrm{C})M}_{LK}\Gamma^{(\mathrm{C})L}_{IJ} = 0 \tag{2.23}$$

where $R^{(\mathrm{C})M}_{IJK}$ is the Riemann–Christoffel tensor and,

$$2\Gamma^{(\mathrm{C})}_{IJK} = C_{JK,I} + C_{KI,J} - C_{IJ,K} \tag{2.24}$$

$$\Gamma^{(\mathrm{C})}_{IJL} = C^{KL}\Gamma_{IJK} \tag{2.25}$$

From symmetry:

$$C_{IJ} = C_{JI} \tag{2.26}$$

$$\Gamma^K_{IJ} = \Gamma^K_{JI} \tag{2.27}$$

$$C_{IJ} = \delta_{IJ} + 2E_{IJ} \tag{2.28}$$

where Γ^K_{IJ} (connection coefficients) is the Christoffel symbol of the second kind;

$$\Gamma^K_{IJ} = \mathbf{G}_{I,J} \cdot \mathbf{G}^K \tag{2.29}$$

$$\bar{\Gamma}^k_{ij} = \mathbf{g}_{i,j} \cdot \mathbf{g}^k \tag{2.30}$$

$$\bar{\Gamma}^K_{IJ} = \mathbf{g}_{I,J} \cdot \mathbf{g}^K \tag{2.31}$$

where G and g (metric tensors) represents co-variant derivatives:

$$\boldsymbol{\nabla} \equiv \frac{\partial}{\partial \mathbf{R}} = \mathbf{G}^I \frac{\partial}{\partial X^I} \tag{2.32}$$

$$\overline{\nabla} \equiv \frac{\partial}{\partial \mathbf{r}} = \mathbf{g}^i \frac{\partial}{\partial x^i} = \mathbf{g}^I \frac{\partial}{\partial X^I} \tag{2.33}$$

and also satisfies that:

$$G_{ij} = \frac{\partial X^\alpha}{\partial x^i}\frac{\partial X^\beta}{\partial x^j} g_{\alpha\beta} \tag{2.34}$$

Taking partial derivative of the metric tensor with respect to x^k:

$$\frac{\partial G_{ij}}{\partial x^k} = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}\frac{\partial X^\beta}{\partial x^j} + \frac{\partial X^\alpha}{\partial x^i}\frac{\partial^2 X^\beta}{\partial x^j \partial x^k}\right) g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}\frac{\partial X^\beta}{\partial x^j}\frac{\partial g_{\alpha\beta}}{\partial x^k} \tag{2.35}$$

and $g_{\alpha\beta} = g_{\beta\alpha}$, material and spatial connection coefficients becomes:

$$(x)\Gamma_{ijk} := \frac{1}{2}\left(\frac{\partial G_{ik}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^k}\right) \tag{2.36}$$

$$(X)\Gamma_{\alpha\beta\gamma} := \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}\right) \tag{2.37}$$

Since there is no curvature in the plane co-variants should be constant and connection coefficients should be zero for the plane strain problem.

Equations of Green-Lagrange strain-displacement:

$$E_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_\alpha}{\partial X_i}\frac{\partial u_\alpha}{\partial X_j}\right] \tag{2.38}$$

In index notation:

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \tag{2.39}$$

Eq. 2.23 satisfies plane-stress large deformation strain field compatibility relations. Using Green's deformation tensor[2]:

$$2E_{KL} \equiv C_{KL}(\mathbf{X}, t) - \delta_{KL} \tag{2.40}$$

$$2E_{xx} = C_{xx} - 1 = 2\frac{\partial U}{\partial X} + \left(\frac{\partial U}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial X}\right)^2 + \left(\frac{\partial W}{\partial X}\right)^2 \tag{2.41}$$

$$2E_{YY} = C_{YY} - 1 = 2\frac{\partial V}{\partial Y} + \left(\frac{\partial U}{\partial Y}\right)^2 + \left(\frac{\partial V}{\partial Y}\right)^2 + \left(\frac{\partial W}{\partial Y}\right)^2 \tag{2.42}$$

$$2E_{ZZ} = C_{ZZ} - 1 = 2\frac{\partial W}{\partial Z} + \left(\frac{\partial U}{\partial Z}\right)^2 + \left(\frac{\partial V}{\partial Z}\right)^2 + \left(\frac{\partial W}{\partial Z}\right)^2 \tag{2.43}$$

$$2E_{XY} = C_{XY} = \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} + \frac{\partial U}{\partial X}\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\frac{\partial V}{\partial Y} + \frac{\partial W}{\partial X}\frac{\partial W}{\partial Y} \tag{2.44}$$

$$2E_{YZ} = C_{YZ} = \frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y} + \frac{\partial U}{\partial Y}\frac{\partial U}{\partial Z} + \frac{\partial V}{\partial Y}\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\frac{\partial W}{\partial Z} \tag{2.45}$$

$$2E_{ZX} = C_{ZX} = \frac{\partial W}{\partial X} + \frac{\partial U}{\partial Z} + \frac{\partial U}{\partial X}\frac{\partial U}{\partial Z} + \frac{\partial V}{\partial X}\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial X}\frac{\partial W}{\partial Z} \tag{2.46}$$

Compatibility conditions for large deformation plane stress:

$$\frac{\partial^2 E_{XX}}{\partial Y^2} - 2\frac{\partial^2 E_{XY}}{\partial x_X \partial x_Y} + \frac{\partial^2 E_{YY}}{\partial X^2} = 0 \tag{2.47}$$

[2]See Eringen

3 Question 3

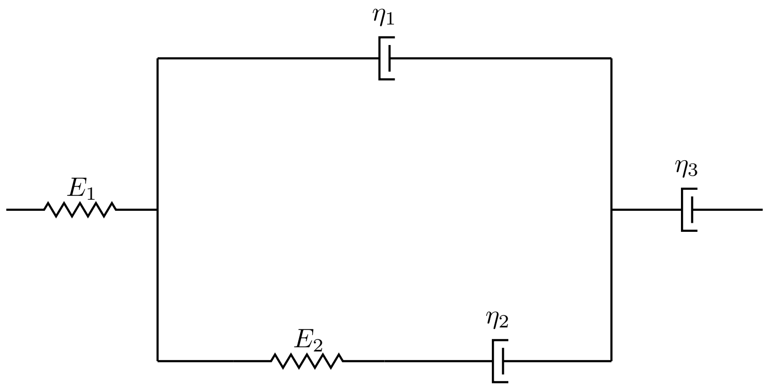

Figure 4: Viscoelastic material model.

Constitutive relations for each elastic and viscous elements within the model in Figure 4;

$$\sigma = E\varepsilon \tag{3.1}$$
$$\sigma = \eta\dot{\varepsilon} \tag{3.2}$$

Relaxation and creep functions of the model are to be obtained.

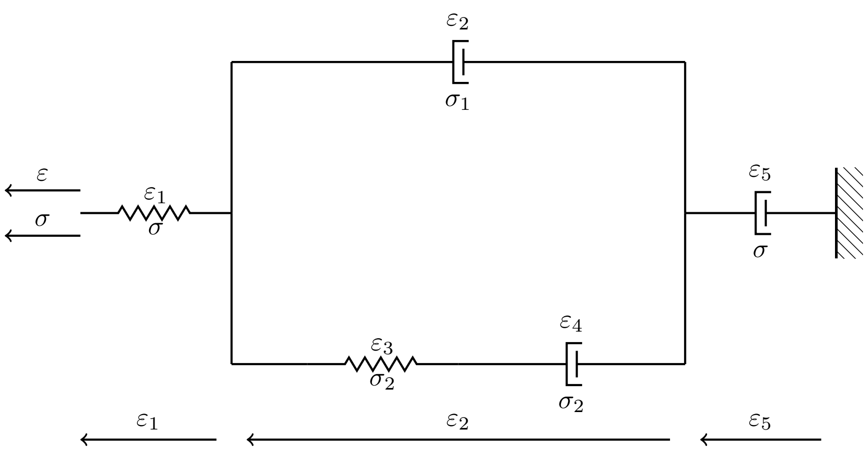

Figure 5: Viscoelastic material model with stresses and strains.

Observations:

$$\varepsilon_2 = \varepsilon_3 + \varepsilon_4 \tag{3.3}$$
$$\sigma_2 = \sigma_3 \tag{3.4}$$
$$\sigma = \sigma_1 + \sigma_2 \tag{3.5}$$
$$\tag{3.6}$$

Constitutive relations:

$$\varepsilon_1 = \frac{\sigma}{E_1} \tag{3.7}$$
$$\varepsilon_5 = \frac{\sigma}{\eta_3} \tag{3.8}$$
$$\dot{\varepsilon_2} = \frac{\sigma_1}{\eta_1} \tag{3.9}$$

For ε_2:

$$\sigma_2 = \eta_2 \dot{\varepsilon_4} + E_2 \varepsilon_3 \tag{3.10}$$
$$\frac{\sigma_2}{\eta_2} = \dot{\varepsilon_4} + \frac{\varepsilon_3 E_2}{\eta_2} \tag{3.11}$$
$$\varepsilon_3 = \frac{\sigma_2}{E_2} \tag{3.12}$$
$$\dot{\varepsilon_4} = \frac{\sigma_2}{\eta_2} \tag{3.13}$$

Using parallel relations:

$$\dot{\varepsilon_2} = \dot{\varepsilon_3} + \dot{\varepsilon_4} \tag{3.14}$$
$$\varepsilon_2 = \varepsilon_3 + \varepsilon_4 \tag{3.15}$$

Also,

$$\varepsilon_2 = \frac{\dot{\sigma_2}}{E_2} + \frac{\sigma_2}{\eta_2} \tag{3.16}$$
$$\sigma_2 = \sigma - \sigma_1 \implies \tag{3.17}$$
$$\frac{\sigma_1}{\eta_1} = \frac{\sigma}{E_2} - \frac{\sigma_1}{E_2} + \frac{\sigma}{\eta_2} + \frac{\sigma_1}{\eta_2} \tag{3.18}$$
$$\sigma(\frac{1}{E_2} + \frac{1}{\eta_2}) = \sigma_1(\frac{1}{E_2} + \frac{1}{\eta_2} + \frac{1}{\eta_1}) \tag{3.19}$$

So,

$$\dot{\varepsilon_2} = \frac{\sigma(\frac{1}{E_2} + \frac{1}{\eta_2} + \frac{1}{\eta_1})}{\eta_1 \frac{1}{E_2} + \frac{1}{\eta_2}} \tag{3.20}$$

Strains:

$$\bar{\varepsilon} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2 + \bar{\varepsilon}_5 \tag{3.21}$$
$$\bar{\varepsilon}_1 = \frac{\bar{\sigma}}{E_1} \tag{3.22}$$
$$(s + \frac{E_2}{\eta_2})\bar{\varepsilon}_2 = \frac{\bar{\sigma_2}}{\eta_2} \tag{3.23}$$
$$s\bar{\varepsilon}_5 = \frac{\bar{\sigma}}{\eta_3} \tag{3.24}$$

Total strain:

$$\bar{\varepsilon} = \frac{\bar{\sigma}}{E_1} + \frac{\bar{\sigma_2}}{\eta_2(s + \frac{E_2}{\eta_2})} + \frac{\bar{\sigma}}{s\eta_3} \tag{3.25}$$

$$\tag{3.26}$$

4 Question 4

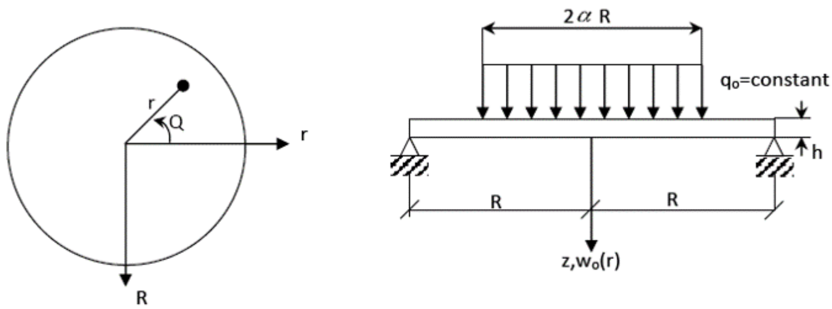

Figure 6: Circular plate with simply supported circumference.

$$w_0(r) = \frac{q_0 R^4}{64D}\left\{\left(\frac{r}{R}\right)^4 + \alpha^2\left[4 - 5\alpha^2 + 4\left(2+\alpha^2\right)\log\alpha\right]\right.$$
$$\left. +2\frac{\alpha^2}{1+v}\left[1-\left(\frac{r}{R}\right)^2\right]\left[4-(1-v)\alpha^2 - 4(1+v)\log\alpha\right]\right\} \tag{4.1}$$

Kirchhoff plate theory yields the solution in Eq. 4.1 for $0 < r \leq \alpha R$ is to be shown. Let's set $\alpha = 1$ and obtain the first part of the problem as shown in 7:

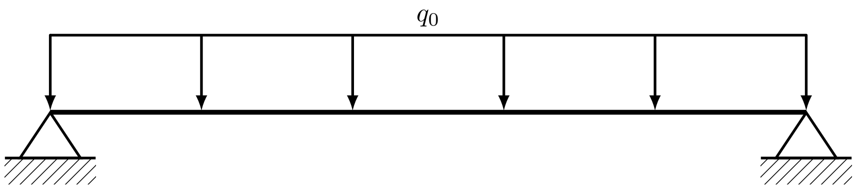

Figure 7: Circular plate with simply supported circumference under axisymmetric load q_0 along $0 < r \leq R$.

For axisymmetric bending and constant material and geometric properties, the equations of equilibrium and stress resultant-displacement relations of the Classical (Kirchhoff) Plate Theory based on the polar coordinate system (r, θ) are summarized below[3].

$$M_r = -D\left(\frac{\mathrm{d}^2 w}{dr^2} + \frac{v}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right) \tag{4.2}$$

$$M_t = -D\left(\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} + v\frac{\mathrm{d}^2 w}{\mathrm{d}r^2}\right) \tag{4.3}$$

$$Q_r = -D\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = -D\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right] \tag{4.4}$$

The differential equation of the deflected surface of the circular plate:

$$\nabla_r^4 w \equiv \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = \frac{q}{D} \tag{4.5}$$

$$\frac{\mathrm{d}^4 w}{\mathrm{d}r^4} + \frac{2}{r}\frac{\mathrm{d}^3 w}{\mathrm{d}r^3} - \frac{1}{r^2}\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r^3}\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{q}{D} \tag{4.6}$$

$\tau_{r\theta} = 0$:

[3] See Wang, Reddy, and Lee; Ventsel, Krauthammer, and Carrera

$$\nabla_r^2 w \equiv \frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right) \tag{4.7}$$

Eq. 4.6 reduces to:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]\right\} = \frac{q}{D} \tag{4.8}$$

Let's w be the solution of this differential equation than:

$$w = w_h + w_p \tag{4.9}$$

where w_h is the homogeneous solution, w_p is the particular solution.
The complementary homogeneous solution of the Eq. 4.8 is given by:

$$w_\mathrm{h} = C_1 \ln r + C_2 r^2 \ln r + C_3 r^2 + C_4 \tag{4.10}$$

Particular solution, w_p, is obtained by integration of Eq. 4.8:

$$w_\mathrm{p} = \int \frac{1}{r} \int r \int \frac{1}{r} \int \frac{r p(r)}{D} \mathrm{d}r\mathrm{d}r\mathrm{d}r\mathrm{d}r \tag{4.11}$$

If $\alpha = 1$:

$$w_\mathrm{p} = \frac{q_0 r^4}{64D} \tag{4.12}$$

So the general solution is:

$$w = C_1 \ln r + C_2 r^2 \ln r + C_3 r^2 + C_4 + \frac{q_0 r^4}{64D} \tag{4.13}$$

Now, lets consider the problem in Figure 6. This problem can be solved using superposition on two different axisymmetric solution for circular plates. The first one is a solid plate with simply supported edge loaded by a line load q around a circle of radius c, as shown in Fig. 8.
To further simplify the solution, let's investigate this plate in two sections. The first one is an annular plate over the region $b \leq r \leq$ which simply supported on the outer edge and have a line force on the inner edge. The second one is a plate with a line force in the outer edges.

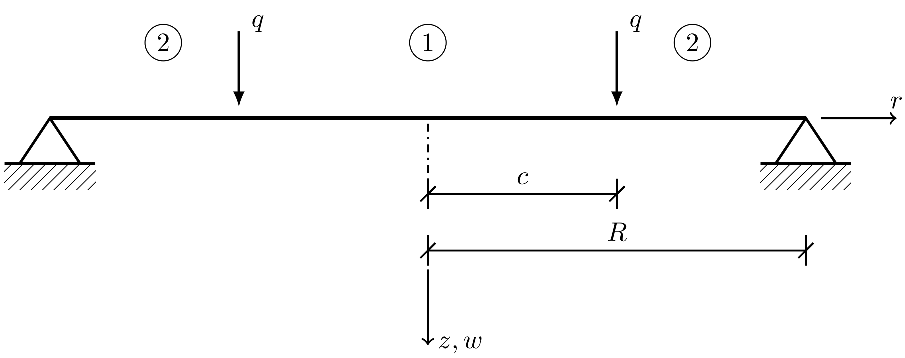

Figure 8: Circular plate with simply supported circumference under line load q along the radius c.

Governing equations in each case:

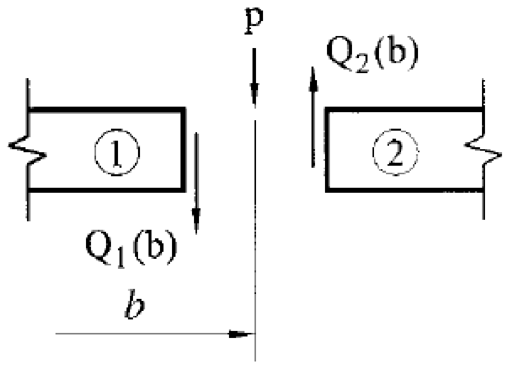

Figure 9: Contact region of plates 1 and 2.

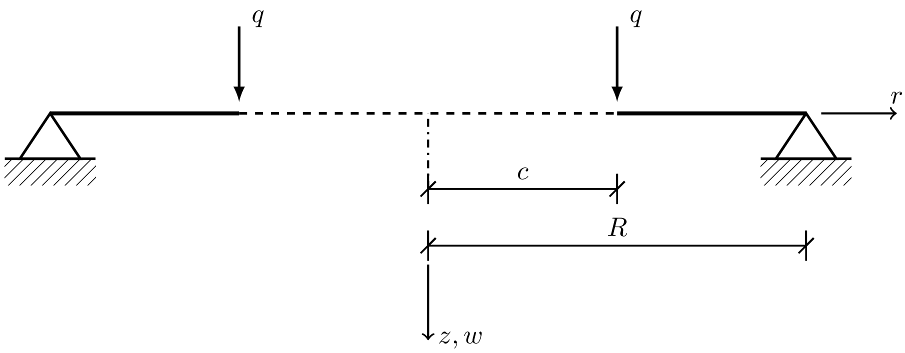

Figure 10: Simply supported plate under a line load uniformly distributed along the inner edge c.

$$\nabla_r^4 w_1 = 0; \nabla_r^4 w_2 = 0 \tag{4.14}$$

General solution for the inner plate (1):

$$w_1 = C_3^{(1)} r^2 + C_4^{(1)} \tag{4.15}$$

General solution for the annular plate (2):

$$w_2 = C_1^{(2)} \ln r + C_2^{(2)} r^2 \ln r + C_3^{(2)} r^2 + C_4^{(2)} \tag{4.16}$$

Continuity conditions at the contact of the two plate segments:

$$w_1|_{r=b} = w_2|_{r=b} \tag{4.17}$$

$$\left.\frac{\mathrm{d}w_1}{\mathrm{d}r}\right|_{r=b} = \left.\frac{\mathrm{d}w_2}{\mathrm{d}r}\right|_{r=b} \tag{4.18}$$

$$\left.M_r^{(1)}\right|_{r=b} = \left.M_r^{(2)}\right|_{r=b} \tag{4.19}$$

$$p = \left.Q_r^{(1)}\right|_{r=b} - \left.Q_r^{(2)}\right|_{r=b} \tag{4.20}$$

Since inner plate is in pure bending:

$$p = -\left.Q_r^{(2)}\right|_{r=b} \tag{4.21}$$

Boundary conditions at $r = R$

$$M_r = 0|_{r=R} \tag{4.22}$$
$$Q_r = -\, p|_{r=R} \tag{4.23}$$

Solution for the annular plate ($c \leq r \leq R$) under a line load uniformly distributed along the inner edge:

$$\begin{aligned} w =& \frac{q_0 R^2 c}{4D} \left\{ \left(1 - \frac{r^2}{R^2}\right) \left[\frac{3+\vartheta}{2(1+\vartheta)} - \frac{c^2}{R^2 - c^2} \ln \frac{c}{R}\right] \right. \\ &\left. + \frac{r^2}{R^2} \ln \frac{r}{R} + \frac{2c^2}{R^2 - c^2} \frac{1+\vartheta}{1-\vartheta} \ln \frac{c}{R} \ln \frac{r}{R} \right\} \end{aligned} \tag{4.24}$$

Solution for the inner plate ($0 \leq r \leq c$):

$$w = \frac{pb}{8a^2 D} \left[\left(a^2 - b^2\right)\left(a^2 + r^2\right) - 2a^2\left(b^2 + r^2\right) \ln \frac{a}{b}\right] \tag{4.25}$$

5 Question 5

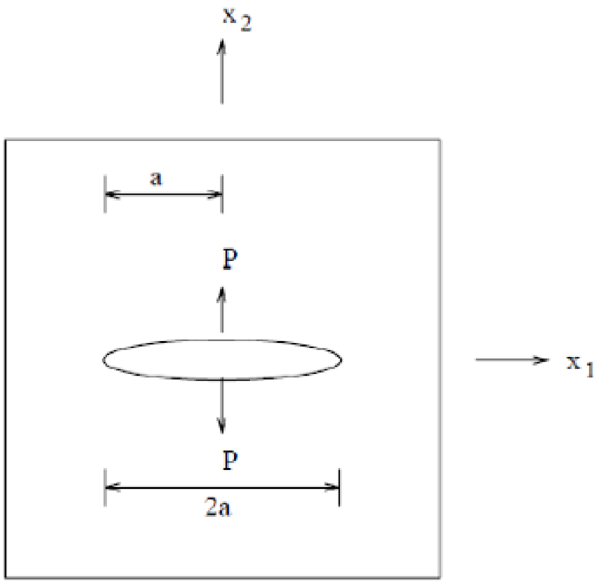

Figure 11: Crack subjected to splitting forces P on the crack surface within an infinite plate

Airy stress function:

$$\phi = \frac{Pa}{\pi z\sqrt{z^2 - a^2}} = \frac{P}{\pi z}\sqrt{\frac{a^2}{z^2 - a^2}} \tag{5.1}$$

where z is a complex number ($z = x_1 + ix_2$) and P is a load per unit thickness.

Assumption 1 - *Shear forces are absent.*

Boundary conditions:

$$\sigma_{22} = 0 \quad \text{at } |x_1| \leq a, x_1 \neq 0 \text{ and } x_2 = 0 \tag{5.2}$$

$$\int_{-a}^{a} \sigma_{22} dx_1 = -P \quad \text{at } x_2 = 0^+ \text{and} \quad x_2 = 0^- \tag{5.3}$$

$$\sigma_{12} = 0 \quad \text{at } |x_1| \leq a \text{ and } x_2 = 0 \tag{5.4}$$

$$\sigma_{11}, \sigma_{22}, \sigma_{12} \to 0 \quad \text{at } x_1^2 + x_2^2 \to \infty \tag{5.5}$$

It can be shown that the Westergaard function for this problem[4]:
Using the Westergaard functions, the following expression for the complex stress intensity factor at the tip $z = a$ can be obtained:

$$K = \sqrt{2\pi} \lim_{z \to a} \left\{ \sqrt{z - a} \left(Z_I - iZ_{II} \right) \right\} \tag{5.6}$$

$$K_I - iK_{II} = \sqrt{2\pi} \lim_{z \to a} \left\{ \sqrt{z - a} \frac{P}{\pi(z - b)} \sqrt{\frac{a^2 - b^2}{z^2 - a^2}} \right\} \tag{5.7}$$

$$= \frac{P}{\sqrt{\pi a}} \sqrt{\frac{a + b}{a - b}} \tag{5.8}$$

[4] See Sun and Jin

The stress intensity factors at the two crack tips:

$$K_I = \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a+b}{a-b}}, \qquad (x = a) \tag{5.9}$$

$$= \frac{P}{\sqrt{\pi a}}\sqrt{\frac{a-b}{a+b}}, \qquad (x = -a) \tag{5.10}$$

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