ME 524 Fracture Mechanics, Homework 2 - Erdem Caliskan

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clear all
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1. Anderson problem 2.12 (section 13.2) (60 Points).

For Single Edge Notched Bend (SE(B)) speciman use S=4W. Note that for Single Edge Notched Tension (SENT) specimen f(a ./ W).* is,

$$f\left(\frac{a}{W}\right)^* = \frac{\sqrt{2\tan\frac{\pi a}{2W}}}{\cos\frac{\pi a}{2W}}\left[0.752 + 2.02\left(\frac{a}{W}\right) + 0.37\left(1 - \sin\frac{\pi a}{2W}\right)^3\right]$$

and for Double Edge Notch Tension (DENT) we have,

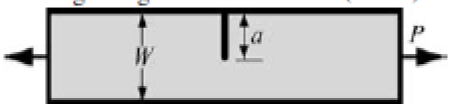
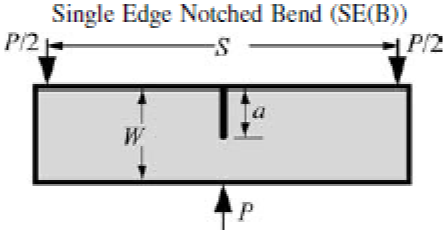
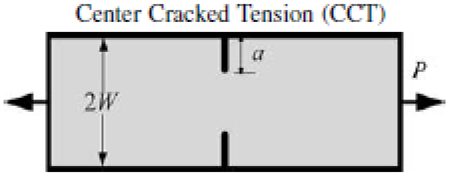
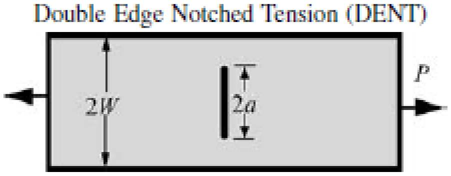
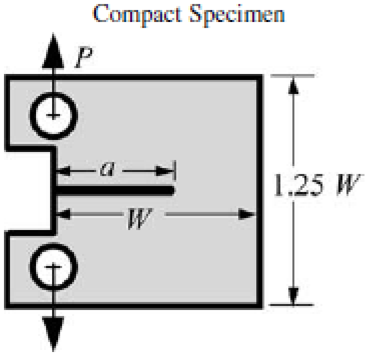
$$f\left(\frac{a}{W}\right)^* = \frac{\sqrt{\frac{\pi a}{W}}}{2\sqrt{1-\frac{a}{W}}}\left[1.122 - 0.561\left(\frac{a}{W}\right) - 0.205\left(\frac{a}{W}\right)^2 + 0.471\left(\frac{a}{W}\right)^3 - 0.190\left(\frac{a}{W}\right)^4\right]$$

Finally, the figures for Center Cracked Tension (CCT) and DENT are flipped (considering the correction in (2), the formulas are correct in the table and you just need to flip the figures).

The handbook H Tada, P.C. Paris, G.R. Irwin, Stress Analysis of Cracks Handbook, 3rd ed., ASME Press. 2000 is a good reference for SIFs, and it is a more reliable source than the Anderson book.

2.12 Consider a material where $K_{IC} = 35 MPa\sqrt{m}\,(31.8\,ksi\sqrt{in.})$. Each of the five specimens in Table 2.4 and Figure 2.23 has been fabricated from this material. In each case, **B = 25.4 mm (1 in.)**, **W = 50.8 mm (2 in.)**, and **a ./ W = 0.5**. Estimate the failure load for each specimen. Which specimen has the highest failure load? Which has the lowest?

TABLE 2.4
K_I Solutions for Common Test Specimens[a]

GEOMETRY	$f\left(\frac{a}{W}\right)^*$
Single Edge Notched Tension (SENT) 	$\frac{\sqrt{2\tan\frac{\pi a}{2W}}}{\cos\frac{\pi a}{2W}}\left[0.752+2.02\left(\frac{a}{W}\right)+0.37\left(1-\sin\frac{\pi a}{2W}\right)^3\right]$
Single Edge Notched Bend (SE(B)) 	$\frac{3\frac{S}{W}\sqrt{\frac{a}{W}}}{2\left(1+2\frac{a}{W}\right)\left(1-\frac{a}{W}\right)^{3/2}}\left[1.99-\frac{a}{W}\left(1-\frac{a}{W}\right)\left\{2.15-3.93\left(\frac{a}{W}\right)+2.7\left(\frac{a}{W}\right)^2\right\}\right]$
Center Cracked Tension (CCT) 	$\sqrt{\frac{\pi a}{4W}\sec\left(\frac{\pi a}{2W}\right)}\left[1-0.025\left(\frac{a}{W}\right)^2+0.06\left(\frac{a}{W}\right)^4\right]$
Double Edge Notched Tension (DENT) 	$\frac{\sqrt{\frac{\pi a}{2W}}}{\sqrt{1-\frac{a}{W}}}\left[1.122-0.561\left(\frac{a}{W}\right)-0.205\left(\frac{a}{W}\right)^2+0.471\left(\frac{a}{W}\right)^3+0.190\left(\frac{a}{W}\right)^4\right]$
Compact Specimen 	$\frac{2+\frac{a}{W}}{\left(1-\frac{a}{W}\right)^{3/2}}\left[0.886+4.64\left(\frac{a}{W}\right)-13.32\left(\frac{a}{W}\right)^2+14.72\left(\frac{a}{W}\right)^3-5.60\left(\frac{a}{W}\right)^4\right]$

$^*K_I = \frac{P}{B\sqrt{W}}f\left(\frac{a}{W}\right)$ where B is the specimen thickness.

[a] Taken from Tada, H., Paris, P.C., and Irwin, G.R., *The Stress Analysis of Cracks Handbook*. 2nd Ed., Paris Productions, St. Louis, MO, 1985.

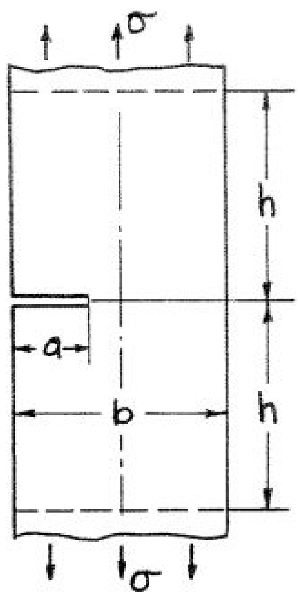
Input

```
B = 25.4*1e-3;
W = 50.8*1e-3;
K_IC = 35*1e6;
a = 0.5*W;
S = 4*W;
```

$$K_I = \frac{P}{B\sqrt{W}} f\left(\frac{a}{W}\right)^*$$

```
calculate_P = @(f) K_IC.*B.*sqrt(W)./f;
```

(a) Single Edge Notched Tension (SENT)



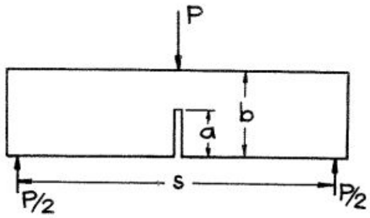
$$f\left(\frac{a}{W}\right)^* = \frac{\sqrt{2\tan\frac{\pi a}{2W}}}{\cos\frac{\pi a}{2W}}\left[0.752 + 2.02\left(\frac{a}{W}\right) + 0.37\left(1 - \sin\frac{\pi a}{2W}\right)^3\right]$$

```
f_SENT = @(a,W) sqrt(2.*tan(pi./2.*a./W))./cos(pi./2.*a./W).*(0.752 + 2.02.*(a./W) ...
    + 0.37.*(1-sin(pi./2.*(a./W))).^3);
f_Single_Edge_Notched_Tension = vpa(f_SENT,4)
```

f_Single_Edge_Notched_Tension =

$$\frac{1.414\sqrt{\tan\left(\frac{1.571\,a}{W}\right)}\left(\frac{2.02\,a}{W} - 0.37\left(\sin\left(\frac{1.571\,a}{W}\right) - 1.0\right)^3 + 0.752\right)}{\cos\left(\frac{1.571\,a}{W}\right)}$$

(b) Single Edge Notched Bending (SENB)



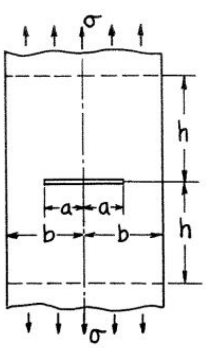
$$f\left(\frac{a}{W}\right)^{*}=\frac{3\,\frac{S}{W}\sqrt{\frac{a}{W}}}{2\left(1+2\frac{a}{W}\right)\left(1-\frac{a}{W}\right)^{3/2}}\left[1.99-\frac{a}{W}\left(1-\frac{a}{W}\right)\left\{2.15-3.93\left(\frac{a}{W}\right)+2.7\left(\frac{a}{W}\right)^{2}\right\}\right]$$

```
f_SENB = @(a,W) 3.*S./W.*sqrt(a./W)./(2.*(1+2.*a./W).*(1-a./W).^(3./2)).*(1.99 ...
    -a./W.*(1-a./W).*(2.15-3.93.*a./W+2.7.*(a./W).^2));
f_Single_Edge_Notched_Bending = vpa(f_SENB,4)
```

f_Single_Edge_Notched_Bending =

$$\frac{0.6096\left(\frac{a\,\left(\frac{a}{W}-1.0\right)\,\left(\frac{2.7\,a^2}{W^2}-\frac{3.93\,a}{W}+2.15\right)}{W}+1.99\right)\sqrt{\frac{a}{W}}}{W\,\left(1.0-\frac{1.0\,a}{W}\right)^{3/2}\,\left(\frac{4.0\,a}{W}+2.0\right)}$$

(c) Center Cracked Tension (CCT)



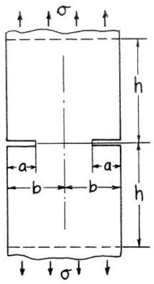
$$f\left(\frac{a}{W}\right)^{*}=\sqrt{\frac{\pi a}{4W}\sec\left(\frac{\pi a}{2W}\right)}\left[1-0.025\left(\frac{a}{W}\right)^{2}+0.06\left(\frac{a}{W}\right)^{4}\right]$$

```
f_CCT  = @(a,W) sqrt(pi./4.*a./W.*sec(pi./2.*a./W)).*(1 - 0.025.*(a./W).^2 ...
    + 0.06.*(a./W).^4);
f_Center_Cracked_Tension = vpa(f_CCT,4)
```

f_Center_Cracked_Tension =

$$1.772\,\left(\frac{0.06\,a^4}{W^4}-\frac{0.025\,a^2}{W^2}+1.0\right)\sqrt{\frac{0.25\,a}{W\cos\left(\frac{1.571\,a}{W}\right)}}$$

(d) Double Edge Notch Tension (DENT)



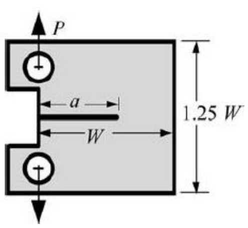
$$f\left(\frac{a}{W}\right)^* = \frac{\sqrt{\frac{\pi a}{W}}}{2\sqrt{1-\frac{a}{W}}}\left[1.122 - 0.561\left(\frac{a}{W}\right) - 0.205\left(\frac{a}{W}\right)^2 + 0.471\left(\frac{a}{W}\right)^3 - 0.190\left(\frac{a}{W}\right)^4\right]$$

```
f_DENT = @(a,W) sqrt(pi.*a./W)./(2.*sqrt(1-a./W)).*(1.122 - 0.561.*(a./W) ...
    - 0.205*(a./W).^2 + 0.471.*(a./W).^3 - 0.190.*(a./W).^4);
f_Double_Edge_Notch_Tension = vpa(f_DENT,4)
```

f_Double_Edge_Notch_Tension =

$$-\frac{0.8862\sqrt{\frac{a}{W}}\left(\frac{0.561\,a}{W} + \frac{0.205\,a^2}{W^2} - \frac{0.471\,a^3}{W^3} + \frac{0.19\,a^4}{W^4} - 1.122\right)}{\sqrt{1.0 - \frac{1.0\,a}{W}}}$$

(e) Compact Specimen



$$f\left(\frac{a}{W}\right)^* = \frac{2+\frac{a}{W}}{\left(1-\frac{a}{W}\right)^{3/2}}\left[0.886 + 4.64\left(\frac{a}{W}\right) - 13.32\left(\frac{a}{W}\right)^2 + 14.72\left(\frac{a}{W}\right)^3 - 5.60\left(\frac{a}{W}\right)^4\right]$$

```
f_CS = @(a,W) (2 + a./W)./(1 - a./W).^(3./2).*(0.886 + 4.64.*(a./W) ...
    - 13.32.*(a./W).^2 + 14.72.*(a./W).^3 - 5.60.*(a./W).^4);
f_Compact_Specimen = vpa(f_CS,4)
```

f_Compact_Specimen =

$$\frac{\left(\frac{a}{W} + 2.0\right)\left(\frac{4.64\,a}{W} - \frac{13.32\,a^2}{W^2} + \frac{14.72\,a^3}{W^3} - \frac{5.6\,a^4}{W^4} + 0.886\right)}{\left(1.0 - \frac{1.0\,a}{W}\right)^{3/2}}$$

Calculations

```
f_single_edge_notched_tension = vpa(f_SENT(a,W),4)
```

f_single_edge_notched_tension = 3.543

```
f_single_edge_notched_bending = vpa(f_SENB(a,W),4)
```

f_single_edge_notched_bending = 10.65

```
f_center_cracked_tension      = vpa(f_CCT(a,W) ,4)
```

f_center_cracked_tension = 0.7434

```
f_double_edge_notch_tension   = vpa(f_DENT(a,W),4)
```

f_double_edge_notch_tension = 0.742

```
f_compact_specimen            = vpa(f_CS(a,W)  ,4)
```

f_compact_specimen = 9.659

```
% P [kN]
P_Single_Edge_Notched_Tension = vpa(calculate_P(f_SENT(a,W))/1e3,4)
```

P_Single_Edge_Notched_Tension = 56.56

```
P_single_edge_notched_bending = vpa(calculate_P(f_SENB(a,W))/1e3,4)
```

P_single_edge_notched_bending = 18.81

```
P_center_cracked_tension      = vpa(calculate_P(f_CCT(a,W) )/1e3,4)
```

P_center_cracked_tension = 269.5

```
P_double_edge_notch_tension   = vpa(calculate_P(f_DENT(a,W))/1e3,4)
```

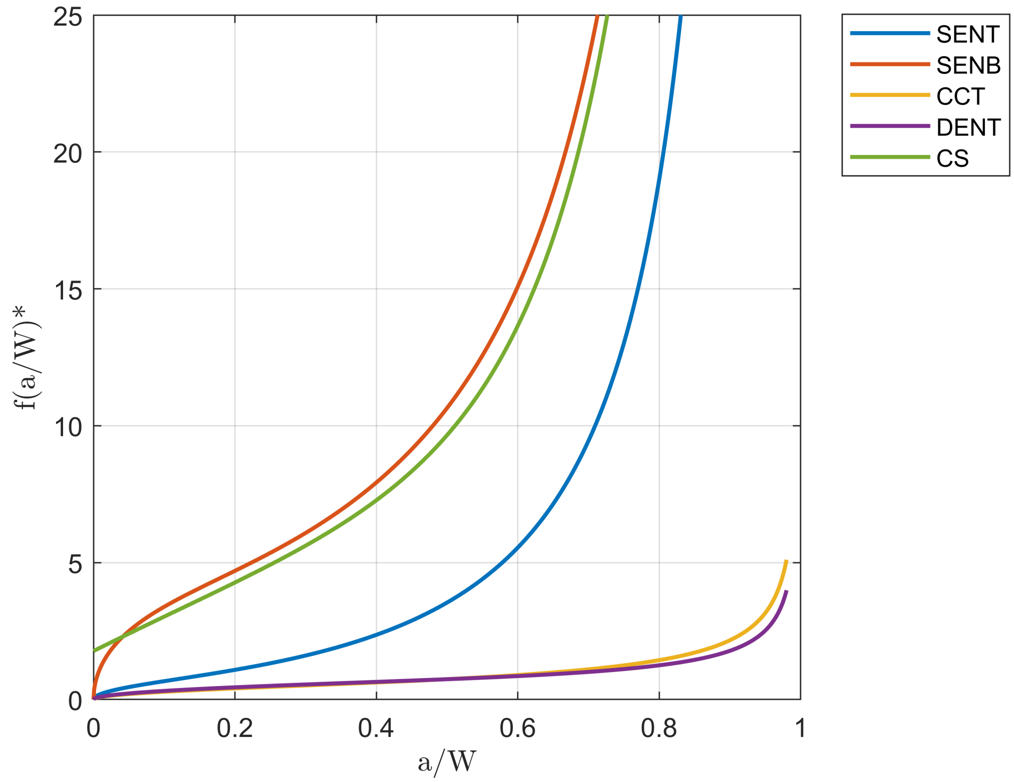
P_double_edge_notch_tension = 270.0

```
P_compact_specimen            = vpa(calculate_P(f_CS(a,W)  )/1e3,4)
```

P_compact_specimen = 20.74

```
a_w = linspace(0,(W-0.02*W),1000);
figure
hold on
plot(a_w/W,f_SENT(a_w,W) ,'LineWidth',1.5)
plot(a_w/W,f_SENB(a_w,W) ,'LineWidth',1.5)
plot(a_w./W,f_CCT(a_w,W) ,'LineWidth',1.5)
plot(a_w./W,f_DENT(a_w,W),'LineWidth',1.5)
plot(a_w./W,f_CS(a_w,W)  ,'LineWidth',1.5)
```

```
legend('SENT','SENB','CCT','DENT','CS',"Location","bestoutside")
box on; grid on; ylim([0 25]); ylabel('f(a/W)*','Interpreter','latex')
xlabel('a/W','Interpreter','latex')
hold off
```



Since $P\,\alpha\,\dfrac{1}{f(a,w)}$ SENT has the lowest critical load, while DENT has the highest critical load.

2. Anderson problem 2.16 (section 13.2). Continuation:

- Compute energy release rate for arbitrary angle β for plane stress condition.

- Assume a constant fracture toughness G_c. Obtain a relation between $\alpha_{\text{ini}},\ \sigma_1,\ \sigma_2,$ and G_C where α_{ini} is the initiation crack length.

- For $\sigma_1 = 2\,\sigma_2$ obtain the angle β_c that corresponds to smallest crack length α_{ini}. Obtain corresponding α_{ini}.

(100 Points)

2.16 Consider a plate subject to biaxial tension with a through crack of length **2a**, oriented at an angle β from the σ_2 axis (Figure 13.1). Derive expressions for K_I and K_{II} for this configuration. What happens to each K expression when $\sigma_1 = \sigma_2$?

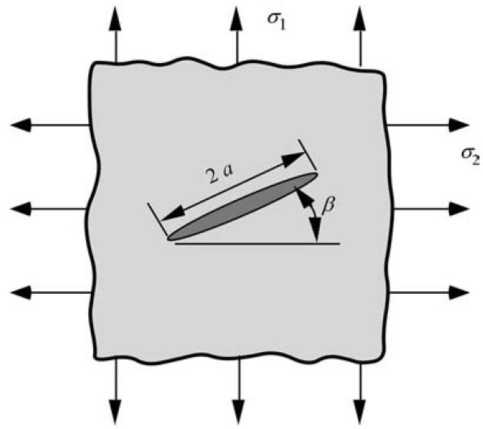


FIGURE 13.1 Through-thickness crack in a biaxially loaded plate (Problem 2.16).

Similar to the figure:

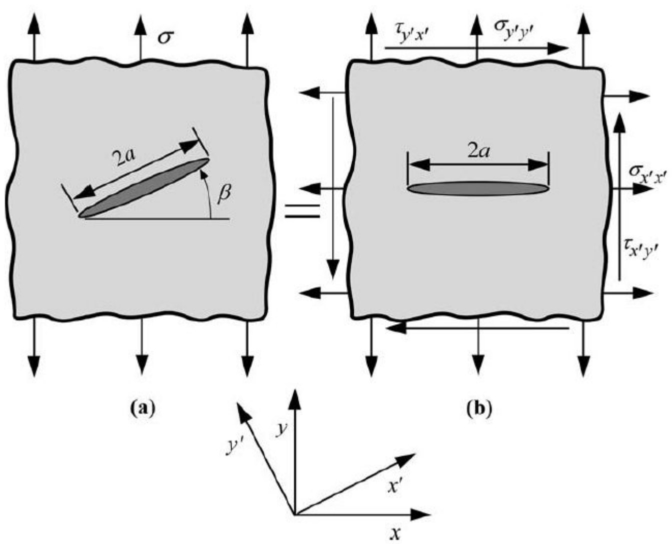


FIGURE 2.18 Through crack in an infinite plate for the general case where the principal stress is not perpendicular to the crack plane.

and the principle of linear superposition: $\sigma_{ij}^{\text{total}} = \sigma_{ij}^{(I)} + \sigma_{ij}^{(\text{II})} + \sigma_{ij}^{(\text{III})}$

$$K_I = \sigma_{y'y'}\sqrt{\pi a} + \sigma_{x'x'}\sqrt{\pi a}$$
$$= \sigma_1 \cos^2(\beta)\sqrt{\pi a} + \sigma_2 \cos^2(\pi/2 + \beta)\sqrt{\pi a}$$
$$K_I = (\sigma_1 \cos^2(\beta) + \sigma_2 \sin^2(\beta))\sqrt{\pi a}$$
$$\sigma_1 = \sigma_2 = \sigma$$
$$K_I = \sigma\sqrt{\pi a}$$

$$K_{II} = \tau_{x'y'}\sqrt{\pi a} + \sigma_{x'x'}\sqrt{\pi a}$$

$$= \sigma_1 \sin(\beta)\cos(\beta)\sqrt{\pi a} + \sigma_2 \sin(\pi/2+\beta)\cos(\pi/2+\beta)\sqrt{\pi a}$$

$$K_{II} = \sigma_1 \sin(\beta)\cos(\beta)\sqrt{\pi a} - \sigma_2 \sin(\beta)\cos(\beta)\sqrt{\pi a}$$

$$K_{II} = (\sigma_1 - \sigma_2)\frac{\sin(\beta/2)\sqrt{\pi a}}{2}$$

$$\sigma_1 = \sigma_2 = \sigma$$

$$K_{II} = 0$$

```
% clear all
% figure
% pos = [0 -1 2 2];
% rectangle('Position',pos,'Curvature',[1 1])
% axis equal
% ylabel('$\tau$','Interpreter','latex')
% xlabel('$\sigma$','Interpreter','latex')
% title(['Mohr Circle for biaxial loading, ' ...
%     '$\sigma_1 = \sigma_2 = \sigma = 1$'],'Interpreter','latex')
```

When all three modes of loading are present, the energy release rate is given by

$$\mathcal{G} = \frac{K_I^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{2\mu}$$

$E = E'$ for plane stress, $K_{\mathrm{III}} = 0$, Thus,

$$\mathcal{G} = (\sigma_1 \cos^2(\beta) + \sigma_2 \sin^2(\beta))^2 \frac{\pi a}{E} + (\sigma_1 - \sigma_2)^2 \sin^2(\frac{\beta}{2}) \frac{\pi a}{4E}$$

$$\mathcal{G} = \mathcal{G}_c \therefore a = a_{ini}$$

When $\mathcal{G} = \mathcal{G}_c$, crack initiates, thus

$$G_C = (\sigma_1 \cos^2(\beta) + \sigma_2 \sin^2(\beta))^2 \frac{\pi a_{ini}}{E} + (\sigma_1 - \sigma_2)^2 \sin^2(\frac{\beta}{2}) \frac{\pi a_{ini}}{4E}$$

```
syms sigma sigma_1 sigma_2 beta a_ini E Gc
K_1 = (sigma_1*cos(beta)^2 + sigma_2*sin(beta)^2)*sqrt(pi*a_ini);
K_2 = ((sigma_1-sigma_2)*cos(beta)*sin(beta))*sqrt(pi*a_ini);
G_c = (K_1^2+K_2^2)/E;
G_c = simplify(G_c)
```

G_c =

$$\frac{\pi\, a_{\mathrm{ini}}\left(-{\sigma_1}^2 \sin(\beta)^2 + {\sigma_1}^2 + {\sigma_2}^2 \sin(\beta)^2\right)}{\mathrm{E}}$$

$$\sigma_1 = 2\sigma_2 = \sigma$$

```
G_c = subs(G_c,[sigma_1 sigma_2],[sigma/2 sigma]);
```

```
G_c = simplify(G_c)
```

G_c =

$$\frac{\pi\, a_{\text{ini}}\, \sigma^2\, (3 \sin(\beta)^2 + 1)}{4\,\text{E}}$$

```
a_ini = isolate(Gc == G_c,a_ini)
```

a_ini =

$$a_{\text{ini}} = \frac{4\,\text{E}\,\text{Gc}}{\sigma^2\, \pi\, (3 \sin(\beta)^2 + 1)}$$

Thus, a_{ini} is largest when $\beta = \frac{\pi}{2},\ 3\frac{\pi}{2}$ and smallest when $\beta = 0,\ 2\pi$

```
subs(a_ini,beta,0) % largest a_ini
```

ans =

$$a_{\text{ini}} = \frac{4\,\text{E}\,\text{Gc}}{\sigma^2\, \pi}$$

2.17 A wide flat plate with a through-thickness crack experiences a nonuniform normal stress that can be represented by the following crack-face traction:

$$p(x) = p_o e^{-x/\beta}$$

where $p_o = 300\,\text{MPa}$ and $\beta = 25\ \text{mm}$. The origin (x=0) is at the left crack tip, as illustrated in Figure 2.27. Using the weight function derived in Example 2.6, calculate K_I at each crack tip for $2a = 25, 50, \text{and } 100\ \text{mm}$. You will need to integrate the weight function numerically.

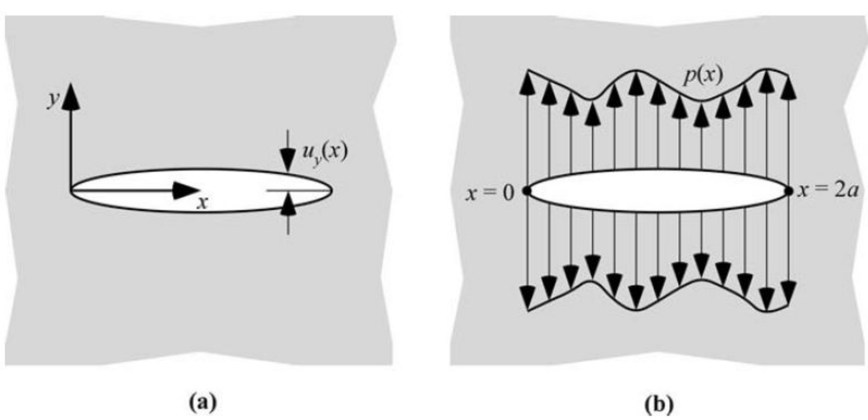


FIGURE 2.27 Through crack configuration analyzed in Example 2.6: (a) definition of coordinate axes and (b) arbitrary traction applied to crack faces.

Uniform tensile stress is applied:

$K_I = \sigma\sqrt{\pi a}$

where a is the half-crack length. From Equation (A2.43), the opening displacement of the crack faces in this case is given by

$$u_y = \pm\frac{2\sigma}{E'}\sqrt{x(2a-x)}$$

where the x-y coordinate axis is defined in Figure 2.27(a). Since the crack length is 2a, we must differentiate u_y with respect to 2a rather than a:

$$\frac{\partial u_y}{\partial(2a)} = \pm\frac{2\sigma}{E}\sqrt{\frac{x}{2a-x}}$$

Thus, the weight function for this crack geometry is given by

$$h(x) = \pm\frac{1}{\sqrt{\pi a}}\sqrt{\frac{x}{2a-x}}$$

If we apply a surface traction of $\pm p(x)$ on the crack faces, the Mode I stress intensity factor for the two crack tips is as follows:

$$K_{I(x=0)} = \frac{1}{\sqrt{\pi a}}\int_0^{2a} p(x)\sqrt{\frac{2a-x}{x}}dx$$

$$K_{I(x=2a)} = \frac{1}{\sqrt{\pi a}}\int_0^{2a} p(x)\sqrt{\frac{x}{2a-x}}dx$$

```
clear all
p_0  = 300*1e6; beta = 25*1e-3;
p = @(x) p_0*exp(-x/beta);
K_x_0  = @(x,a) (p(x).*sqrt((2*a-x)/x));
K_x_2a = @(x,a) (p(x).*sqrt(x/(2*a-x)));
for a = [25 50 100]/2*1e-3
    fprintf(1,'For 2a = %.1f mm',2*a*1e3)
    K_I_0  = 1./sqrt(pi*a)*integral(@(x) K_x_0 (x,2*a),0,2*a);
    K_I_2a = 1./sqrt(pi*a)*integral(@(x) K_x_2a(x,2*a),0,2*a);
    fprintf(1,'K_I_@(0)  = %.3f MPa*(m^-1/2)',K_I_0 /1e6)
    fprintf(1,'K_I_@(2a) = %.3f MPa*(m^-1/2)',K_I_2a/1e6)
end
```

```
For 2a = 25.0 mm
K_I_@(0)  = 31.122 MPa*(m^-1/2)
K_I_@(2a) = 12.317 MPa*(m^-1/2)
For 2a = 50.0 mm
K_I_@(0)  = 30.103 MPa*(m^-1/2)
K_I_@(2a) = 11.913 MPa*(m^-1/2)
For 2a = 100.0 mm
K_I_@(0)  = 24.166 MPa*(m^-1/2)
K_I_@(2a) = 9.564 MPa*(m^-1/2)
```

2.20 A material has a yield strength of 345 MPa (50 ksi) and a fracture toughness of 110 MPa (100 ksi .). Determine the minimum specimen dimensions (B, a, W) required to perform a valid KIc test on this material, according to ASTM E 399. Comment on the feasibility of testing a specimen of this size.

7. Specimen Size, Configurations, and Preparation

7.1 *Specimen Size:*

7.1.1 In order for a result to be considered valid according to this test method (see also 3.1.2.1), the specimen ligament size $(W - a)$ must be not less than $2.5(K_{Ic}/\sigma_{YS})^2$, where σ_{YS} is the 0.2 % offset yield strength of the material in the environment and orientation, and at the temperature and loading rate of the test (**1, 3, 4**). For testing at rates other than quasi-static see Annex A10, Rapid Force Testing. The specimen must also be of sufficient thickness, B, to satisfy the specimen proportions in 7.2.1 or 7.2.1.1 and meet the P_{max}/P_Q requirement in 9.1.3. Meeting the ligament size and P_{max}/P_Q requirements cannot be assured in advance. Thus, specimen dimensions shall be conservatively selected for the first test in a series. If the form of the material available is such that it is not possible to obtain a test specimen with ligament size equal to or greater than $2.5(K_{Ic}/\sigma_{YS})^2$, then it is not possible to make a valid K_{Ic} measurement according to this test method.

```
clear all
KIc      = 110;
sigma_ys = 345;
W_a = 2.5*(KIc/sigma_ys)^2;
fprintf(1,'W - a > %f mm',W_a*1e3)
```

```
W - a > 254.148288 mm
```

7.2.1 *Specimen Proportions*—Crack size, a, is nominally between 0.45 and 0.55 times the width, W. Bend specimens can have a width to thickness, W/B, ratio of $1 \leq W/B \leq 4$. Tension specimen configurations can be $2 \leq W/B \leq 4$.

Let $W - a = 2.5 * (K_{IC}/\sigma_{YS})^2$, $a = 0.45W$, thus $0.55W = 2.5 * (K_{IC}/\sigma_{YS})^2$, and $W < 4B$

```
W = W_a/0.55*1e3 % mm
```

```
W = 462.0878
```

```
a = 0.45*W
```

```
a = 207.9395
```

```
B = W/4
```

```
B = 115.5219
```

The required dimensions are too large to produce within a reasonable budget, considering the required surface finish standards. The material is too tough. Thus, it is not possible to make a valid KIc measurement according to this test method.