Erdem Caliskan HW3

12.2 A two-dimensional plane strain finite element analysis is performed on a through crack in a wide plate (Figure 2.3). The remote stress is 100 MPa, and the half-crack length = 25 mm. The stress normal to the crack plane (σ_{yy}) at $\theta = 0$ is determined at node points near the crack tip and is tabulated in the given table. Estimate K_I by means of the stress-matching approach (Equation (12.14)) and compare your estimate to the exact solution for this geometry. Is the mesh refinement sufficient to obtain an accurate solution in this case?

$\frac{r}{a}(q=0)$	$\frac{\sigma_{yy}}{\sigma^\infty}$	$\frac{r}{a}(q=0)$	$\frac{\sigma_{yy}}{\sigma^\infty}$
0.005	11.0	0.080	3.50
0.010	8.07	0.100	3.24
0.020	6.00	0.150	2.83
0.040	4.54	0.200	2.58
0.060	3.89	0.250	2.41

$$K_I = \lim_{r\to 0}[\sigma_{yy}\sqrt{2\pi r}] \quad (\theta = 0)$$

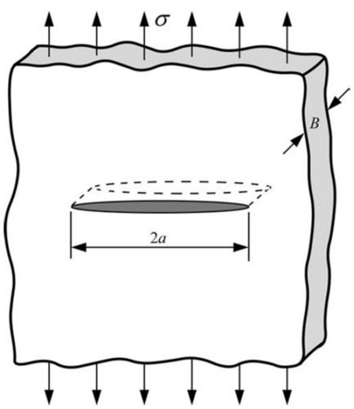


FIGURE 2.3 A through-thickness crack in an infinitely wide plate subjected to a remote tensile stress.

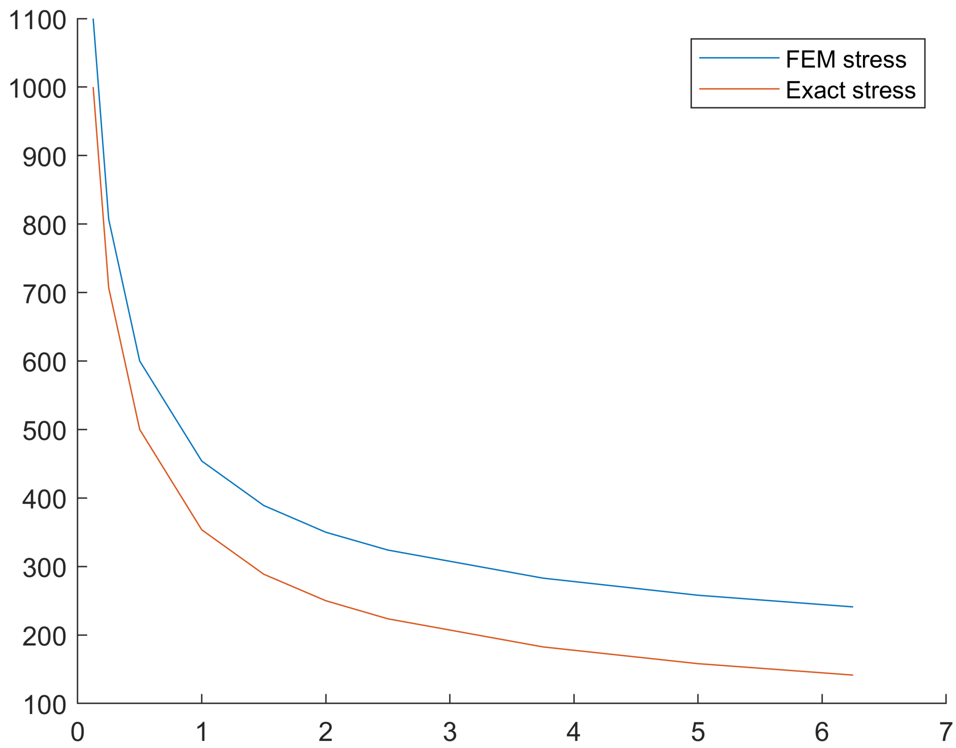
```
K_exact = 100*sqrt(pi*25)
```

```
K_exact = 886.2269
```

```
r    =  25*[0.25 0.20 0.15 0.10 0.08 0.06 0.04 0.02 0.01 0.005];
s_yy = 100*[2.41 2.58 2.83 3.24 3.50 3.89 4.54 6.00 8.07 11   ];
interp1(r,s_yy.*sqrt(2*pi.*r),0,'linear','extrap');
fprintf(1,'K_I = %.2f',ans);
```

```
K_I = 938.27
```

```
figure
hold on
plot(r,s_yy)
plot(r,K_exact./sqrt(2*pi.*r))
legend('FEM stress','Exact stress')
```



Refining the near the crack-tip would improve the solution. One can also use singular elements to get more accurate results.

12.3 Displacements at nodes along the upper crack face (u_y at $\theta = \pi$) in the previous problem are tabulated in the given table. The elastic constants are as follows: $E = 208{,}000$ MPa and $\nu = 0.3$. Estimate K_I by means of the (plane strain) displacement-matching approach (Equation (12.15a)) and compare your estimate to the exact solution for this geometry. Is the mesh refinement sufficient to obtain an accurate solution in this case?

$\frac{r}{a}(\theta = \pi)$	$\frac{u_y}{a}$	$\frac{r}{a}(\theta = \pi)$	$\frac{u_y}{a}$
0.005	9.99×10^{-5}	0.080	3.92×10^{-4}
0.010	1.41×10^{-4}	0.100	4.36×10^{-4}
0.020	1.99×10^{-4}	0.150	5.27×10^{-4}
0.040	2.80×10^{-4}	0.200	6.00×10^{-4}
0.060	3.41×10^{-4}	0.250	6.61×10^{-4}

$$K_I = \lim_{r \to 0} \left[\frac{E u_y}{4(1-\nu^2)} \sqrt{\frac{2\pi}{r}} \right] \quad (\theta = \pi)$$

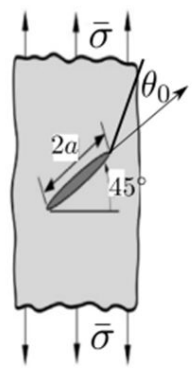
```
E  = 208000;
```

```
nu = 0.3;
r    =     25*[0.25 0.20 0.15 0.10 0.08 0.06 0.04 0.02 0.01 0.005];
u_y  =1e-4*25*[6.61 6.00 5.27 4.36 3.92 3.41 2.80 1.99 1.41 0.999];
interp1(r,E.*u_y/4/(1-nu^2).*sqrt(2*pi./r),0,'linear','extrap');
fprintf(1,'K_I = %.2f',ans)
```

```
K_I = 1013.82
```

Mesh refiniment would increase the accuracy. Ideally, displacement field can get very close since it has the same form with the exact solution.

3. In figure below, a crack with initial angle of 45° is under uniaxial far field loading σ_0 in an sufficiently large domain (*e.g.*, infinite domain SIF formula can be used).



- Using maximum circumferential tensile stress obtain the angle θ_0 (angle relative to original crack direction not the horizontal x axis) at which the crack would propagate. Hint: Refer to Saoma notes pages 160-161.
- For a given fracture toughness K_{Ic} we can express the maximum traction $\bar{\sigma}_{\max}$ for which the crack would not propagate using maximum circumferential tensile stress criteria. Express $\bar{\sigma}_{\max} = \alpha_{\text{MCTS}} \frac{K_{Ic}}{\sqrt{a}}$ for a nondimensional value α_{MCTS}.
- Compare θ_0 you obtained with figure 10.4 in Saouma notes (p. 165/446) for the problem in figure 3 ($K_I = K_{II}$). How is this θ_0 compared with θ_0 obtained from maximum energy release rate and minimum strain energy density criteria shown in the same figure?
- Referring to figure 10.5 in Saoma notes (p.165/446) compare the traction $\bar{\sigma}_{\max}$ that would initiate crack propagation in terms of nondimensional parameter $\alpha = \frac{\bar{\sigma}_{\max}\sqrt{a}}{K_{Ic}}$ based on maximum energy release rate (α_{MERR}) and minimum strain energy density (α_{MSED}). Again, limit your discussion to the problem in figure 3 ($K_I = K_{II}$). Which one is the most conservative and which one is the lease conservative? Note that numerical values of all three α coefficients are needed (use the figure to obtain K_I/K_{I_c} and by expressing K_I in terms of $\bar{\sigma}_{\max}\sqrt{a}$ find the value of α).

(**60 Points**)

σ_θ **reaches maximum when** $\tau_{r\theta} = 0$

Pure mode I loading:

$$\sigma_r = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1+\sin^2\frac{\theta}{2}\right) \quad (6.59\text{-a})$$

$$\sigma_\theta = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right) \quad (6.59\text{-b})$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos^2\frac{\theta}{2} \quad (6.59\text{-c})$$

Pure mode II loading:

$$\sigma_r = \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{5}{4}\sin\frac{\theta}{2}+\frac{3}{4}\sin\frac{3\theta}{2}\right) \quad (6.60\text{-a})$$

$$\sigma_\theta = \frac{K_{II}}{\sqrt{2\pi r}} \left(-\frac{3}{4}\sin\frac{\theta}{2}-\frac{3}{4}\sin\frac{3\theta}{2}\right) \quad (6.60\text{-b})$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{1}{4}\cos\frac{\theta}{2}+\frac{3}{4}\cos\frac{3\theta}{2}\right) \quad (6.60\text{-c})$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2}+\frac{3}{4}\cos\frac{3\theta}{2}\right)$$

$$\cos\frac{\theta_0}{2}\left[K_I \sin\theta_0 + K_{II}(3\cos\theta_0 - 1)\right] = 0$$

```
clear all

syms K_I K_II 'positive'
syms theta_0

eqn = cos(theta_0/2)*(K_I*sin(theta_0) + K_II*(3*cos(theta_0)-1))
```

eqn =

$$\cos\left(\frac{\theta_0}{2}\right)\ (K_{II}\ (3\cos(\theta_0)-1)+K_I\sin(\theta_0))$$

```
vars = theta_0;
assume(theta_0 ~= pi);
sltn= solve(eqn,vars,"Real",true,"ReturnConditions",true,"IgnoreAnalyticConstraints",true);
sltn.theta_0
```

ans =

$$\begin{pmatrix} 2\,\mathrm{atan}\left(\frac{K_I-\sqrt{{K_I}^2+8\,{K_{II}}^2}}{4\,K_{II}}\right) \\ 2\,\mathrm{atan}\left(\frac{K_I+\sqrt{{K_I}^2+8\,{K_{II}}^2}}{4\,K_{II}}\right) \end{pmatrix}$$

$$\sigma_{\theta\,\max}\sqrt{2\pi a} = K_{Ic} = \cos\frac{\theta_0}{2}\left[K_I\cos^2\frac{\theta_0}{2} - \frac{3}{2}K_{II}\sin\theta_0\right]$$

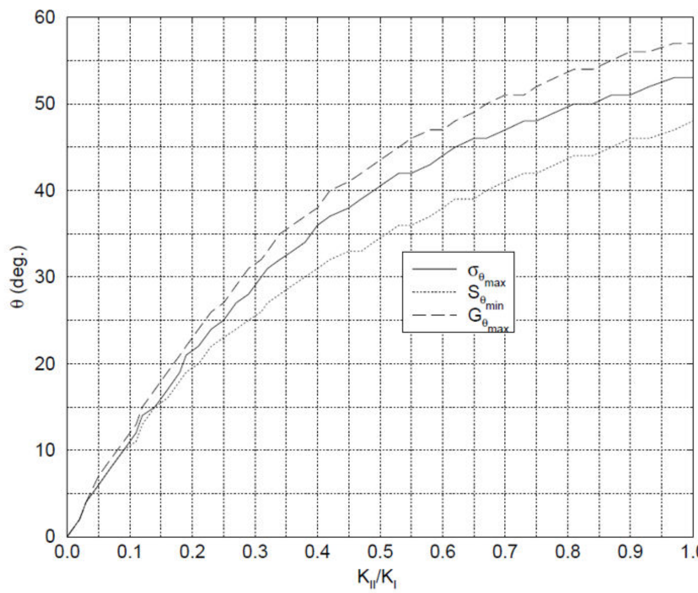
$$\bar{\sigma}_{\theta\,\max} = \alpha_{MCTS}\frac{K_{Ic}}{\sqrt{a}} = \frac{1}{\sqrt{2\pi}}\frac{K_{Ic}}{\sqrt{a}}$$

```
assumeAlso(K_I == K_II)
```

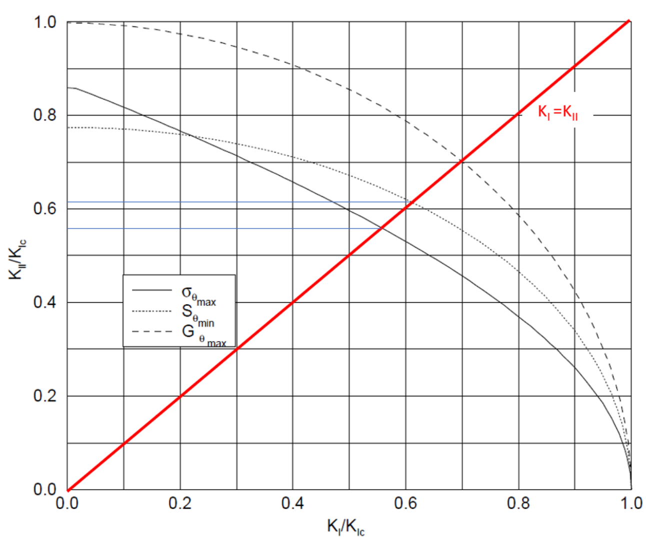
```
simplifiedExpr = simplify(sltn.theta_0,"Steps",1600);
vpa((subs(simplifiedExpr,sltn.parameters,0)*180/pi),5)
```

ans =

$$\begin{pmatrix} -53.13 \\ 90.0 \end{pmatrix}$$



We found $\theta_0 = 53.13$ which match with the figure. Minimum strain energy density yields a higher angle while the maximum energy release rate yields lower. Maximum circumferential tensile stress method gives us a in-between angle value.



$$\alpha = \frac{\bar{\sigma}_{\max} \sqrt{a}}{K_{IC}}$$

Maximum circumferential tensile stress: 0.56 :: Least conservative

Minimum strain energy density: 0.61

Maximum energy release rate: 0.70 :: Most conservative

4. A crack growth at a rate $(\frac{da}{dN})_1 = 8.84 \times 10^{-7} \frac{\text{m}}{\text{cycle}}$ when the stress intensity factor is $(\Delta K)_1 = 50\ \text{MPa}\sqrt{\text{m}}$ and at a rate $(\frac{da}{dN})_1 = 4.13 \times 10^{-5} \frac{\text{m}}{\text{cycle}}$ when $(\Delta K)_2 = 150\ \text{MPa}\sqrt{\text{m}}$. Determine the parameters C and m in Paris equation. (**60 Points**)

$$\frac{da}{dN} = C\Delta K^m$$

```
syms m C

eqns = [50^m*C == 8.84*1e-7, 150^m*C == 4.13*1e-5];
vars = [m C];
[m, C] = solve(eqns,vars);
m = vpa(solm,4)
```

m = 3.499

```
C = vpa(solC,4)
```

C = 1.004e-12