ME 524 Midterm - Erdem Caliskan

Question 1

1. (90 Points). A cylindrical pressure vessel with closed ends has a radius R = 1 m and thickness t = 40 mm and is subjected to internal pressure p. The vessel must be designed safely against failure by yielding (according to the von Mises yield criterion) and fracture. The vessel is made of steel with yield stress σy = 860 MPa and fracture toughness KIc = 100 MPa√ m.

```
clear all
R = 1      ;    % m
T = 40*1e-3;    % m
Sigma_ys = 860; % MPa
KIc      = 100; % MPa*sqrt(m)
```

(a) For von Mises yield stress, yielding occurs when, $\sigma_v = \sigma_y \quad \text{for} \quad \sigma_v = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$

where σ1, σ2, σ3 are principal stresses. By using the values of σ_{zz}, $\sigma_{\theta\theta}$, and $\sigma_{rr} = 0$ (exterior surface of the vessel, producing largest σ_v), obtain p_p the maximum allowable p from plastic yielding perspective.

$$\sigma_{\theta\theta} = \frac{pR}{t}$$

$$\sigma_{zz} = \frac{pR}{2t}$$

$$\sigma_{rr} = 0$$

```
syms p r t sigma_ys
sigma_tt = p*r/t; sigma_zz = p*r/(2*t);
sigma_v = sqrt(((sigma_tt-sigma_zz)^2+(sigma_tt)^2+(sigma_tt)^2)/2)
```

sigma_v =

$$\sqrt{\frac{9\,p^2\,r^2}{8\,t^2}}$$

```
p = isolate(sigma_v == sigma_ys,p)
```

p =

$$p = \frac{2\,\sqrt{2}\,\sigma_{ys}\,t}{3\,r}$$

```
P = subs(p,[sigma_ys t r],[Sigma_ys T R]);
vpa(P,4) % MPa
```

ans = $p = 32.43$

(b) What direction of the crack between axial to circumferential direction experiences the highest stress intensity factor?

A crack oriented with the axial direction experiences the highest stress intensity factor since the hoop stress is two times larger than the longitudinal stress.

c) Plot the maximum permissible pressure p_c versus crack length a_c considering both plastic yielding and fracture. Employ LEFM model for fracture analysis. To simplify the problem, consider the crack in the worst direction for fracture for both fracture and plastic yielding consideration. The crack is through thickness. Finally, the (axial) length of the cylinder is assumed to be much larger than crack length. So, based on the information provided you may not need to decrease plastic yielding untimate stress based on the reduction of remaining area.

$$K_I = \sigma\sqrt{\pi a}$$

$$\sigma = \frac{K_I}{\sqrt{\pi a}}$$

$$\frac{pR}{t} = \frac{K_I}{\sqrt{\pi a}}$$

$$p = \frac{K_I t}{R\sqrt{\pi a}}$$

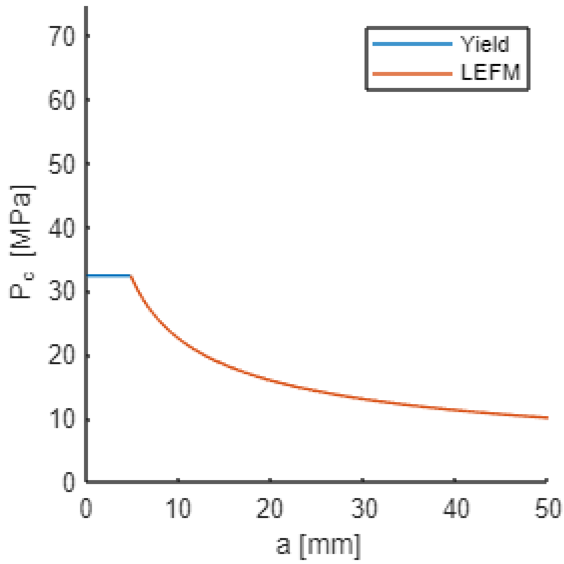
```
P_c_plastic = double(solve(P))
```

```
P_c_plastic = 32.4326
```

```
a = linspace(0.1,50,1000)*1e-3;

P_c_plastic = P_c_plastic*ones(size(a));
P_c_lefm = KIc*T./(R*sqrt(pi*a));

for i = 1:length(a)
    if P_c_lefm(i) >= P_c_plastic
        P_c_lefm(i) = nan;
        a_cr = a(i);
    else
        P_c_plastic(i) = nan;
    end
end
figure('Position',[100 100 300 300])
hold on
plot(a*1e3,P_c_plastic)
plot(a*1e3,P_c_lefm)
ylim([0 75]); xlim([0 max(a)*1e3]); legend('Yield', 'LEFM');
xlabel('a [mm]'); ylabel('P_c [MPa]')
```



(d) What is the crack length a_{tran} corresponding to the transition between plastic and fracture failure mechanisms?

$$P_{c_{plastic}} = \frac{K_I t}{R\sqrt{\pi a}}$$

$$\sqrt{\pi a} = \frac{K_I t}{R P_{c_{plastic}}}$$

$$\pi a = \left(\frac{K_I t}{R P_{c_{plastic}}}\right)^2$$

```
% approximately
a_cr = a_cr*1e3 % mm
```

```
a_cr = 4.7953
```

```
% analytically
a_cr = (KIc*T/(R*P_c_plastic(1)))^2/pi*1e3 % mm
```

```
a_cr = 4.8418
```

$a_{tran} = 4.8418$ **is the crack length corresponding to transition between plastic and fracture failure mechanisms.**

(e) Calculate the maximum permissible crack length a_c for an operating pressure $p = 12\ MPa$.

```
a_c = (KIc*T/(R*12))^2/pi*1e3 % mm
```

```
a_c = 35.3678
```

(f) Calculate the failure pressure p_c for a minimum detectable crack length $a = 1\ cm$.

$a = 1\ cm$ **falls into LEFM dominated zone, thus**

$p = \frac{K_I t}{R\sqrt{\pi a}}$

```
p_c_10 = KIc*T./(R*sqrt(pi*0.1))
```

```
p_c_10 = 7.1365
```

(g) Calculate the failure pressure p_c for a minimum detectable crack length $a = 1\,mm$.

$a = 1\,mm$ **falls into yield dominated zone, thus**

```
p_c_1 = P_c_plastic(1)
```

```
p_c_1 = 32.4326
```

Question 2

2. (60 Points). For the notch problem shown in (1) we obtain the power of singularity for strain and strain $\left(\lambda_1 - 1 = -\frac{1}{2}\right)$ from the equation $\sin(2\pi\lambda_n) = 0 \quad \Rightarrow \quad \lambda_n = \frac{n}{2}, n > 1$. Using the equation,

$$\sin(2\lambda\alpha) + \lambda\sin(2\alpha) = 0 \quad \text{mode I}$$
$$\sin(2\lambda\alpha) - \lambda\sin(2\alpha) = 0 \quad \text{mode II}$$

obtain the power of singularity of stress and strain $(\lambda - 1)$ for modes I and II. To ensure that internal energy is finite around the crack tip $\lambda - 1 \geq -\frac{1}{2}(U\mathrm{dA} = \sigma\epsilon r \ \ \mathrm{dr d\theta}\ \text{bounded for}\ r \to 0)$. Also, for the singular response $\lambda - 1 < 0$. So the acceptable range for the first term λ is $\frac{1}{2} \leq \lambda < 1$ for a singular response. For more information refer to the course presentation pages 135-138.

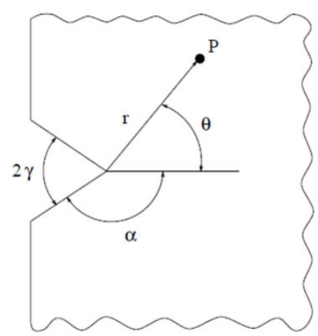


Figure 1: Schematic of notch geometry

```
clear all
% syms lambda alpha x y
a = pi;
mode_1 = @(lambda) sin(2*lambda*a) + lambda*sin(2*a);
mode_2 = @(lambda) sin(2*lambda*a) - lambda*sin(2*a);
lambda_mode_1 =fzero(mode_1,0.5)
```

```
lambda_mode_1 = 0.5000
```

```
lambda_mode_2 =fzero(mode_2,0.5)
```

```
lambda_mode_2 = 0.5000
```

- Find the stress and strain singularity power of mode I and II for 90° notch $\alpha = \frac{3}{4}\pi$. You need to obtain the λ_1 the minimum root of equations (2) for $\lambda \in \left[\frac{1}{2}\ 1\right)$.

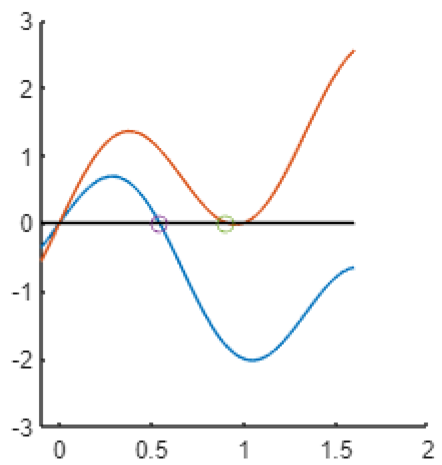
```
a = 3/4*pi;
mode_1 = @(lambda) sin(2*lambda*a) + lambda*sin(2*a);
mode_2 = @(lambda) sin(2*lambda*a) - lambda*sin(2*a);
lambda_mode_1 =fzero(mode_1,0.75)
```

```
lambda_mode_1 = 0.5445
```

```
lambda_mode_2 =fzero(mode_2,0.75)
```

```
lambda_mode_2 = 0.9085
```

```
x_min = -0.1;
x_max = 1.6;
figure ('Position',[100 100 250 250])
hold on
fplot(mode_1,[x_min x_max])
fplot(mode_2,[x_min x_max])
plot(x_min:0.1:x_max,(x_min:0.1:x_max)*0,'Color','black')
scatter(lambda_mode_1,0)
scatter(lambda_mode_2,0)
hold off
```




```
fprintf('Power of singularity for Mode 1 (lambda - 1) = %f',lambda_mode_1-1)
```

```
Power of singularity for Mode 1 (lambda - 1) = -0.455516
```

```
fprintf('Power of singularity for Mode 2 (lambda - 1) = %f',lambda_mode_2-1)
```

```
Power of singularity for Mode 2 (lambda - 1) = -0.091471
```

- Noting that $\sigma = K_I r^{\lambda^I - 1} + K_{II} r^{\lambda^{II} - 1} + \ldots$ discuss which mode will dominate the stress field near the crack tip.

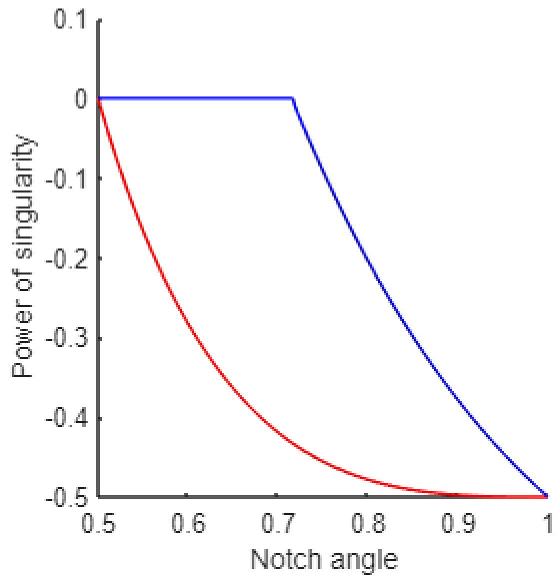
How is this compared to sharp crack, $\alpha = \pi$, where $\lambda^I = \lambda^{II} = \frac{1}{2}$.

Power of singularities are $(\lambda - 1) = -0.455516$ for Mode 1 and $(\lambda - 1) = -0.091471$ for Mode 2. Thus, Mode 1 will dominate the stress field around the crack tip.

- For your interest, no need to submit. Plot radius of singularity (when applicable) for modes I and II for $\alpha = \pi/2$ to π.

```
i = 1;
for a = pi/2:0.01:pi
    mode_1 = @(lambda) sin(2*lambda*a) + lambda*sin(2*a);
    mode_2 = @(lambda) sin(2*lambda*a) - lambda*sin(2*a);
    lambda_mode_1(i) = fzero(mode_1,0.75);
    lambda_mode_2(i) = fzero(mode_2,0.75);
    if lambda_mode_2(i) < 1e-6
        lambda_mode_2(i) = fzero(mode_2,1);
    end
    alpha(i) = a;
    i = i+1;
end

figure('Position',[100 100 300 300])
hold on
plot(alpha/pi,lambda_mode_1-1,'Color','red')
plot(alpha/pi,lambda_mode_2-1,'Color','blue')
hold off
xlim([0.5 1])
ylabel('Power of singularity')
xlabel('Notch angle')
```



Response gets more singular as the crack sharpens. Mode 2 response is not singlular for notch angles ~<130 degrees.

Question 3

3. (150 Points). Figure 2 shows a point force displacement system with crack length A, force P, and beam width and height B and $2\mathrm{H}$, respectively. The moment at the end of the crack due to the force is $M = PA$. To distinguish A from area of the crack surface we use $\mathscr{A} = AB$ for the latter. We employ the following nondimensional parameters to facilitate the analysis of this problem,

$$
\begin{aligned}
a &= \frac{A}{H} && \text{normalized crack length} \\
p &= \frac{P}{\sigma_y B H} && \text{normalized force} \\
m &= pa = \frac{PA}{\sigma_y B H^2} && \text{normalized moment} \\
\delta &= \frac{\Delta}{H}\frac{E}{\sigma_y} && \text{normalized displacement (crack opening)}
\end{aligned}
$$

where σ_y is the yield stress.

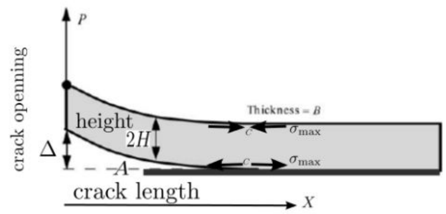


Figure 2: Force displacement relation for a point force system.

The purpose of this problem is plastic fracture mechanics analysis of this crack and comparison with LEFM. We adapt an elastic-perfectly plastic material behavior. From linear analysis we know that the maximum moment M that this beam can withstand without plastic deformation is when σ_{max} at points C in the figure reach σ_y. If M further increases (through increasing P or crack length A) we will have plastic yielding in points C and the plastic region further penetrates inside the domain, until M at crack tip eventually reaches maximum possible moment that the section can withstand. The limit for initiation of plastic deformation and maximum value moments are,

$$\begin{aligned} M_{\mathrm{lmax}} &= \frac{I}{\sigma_y} z_{\mathrm{max}} = \frac{2}{3} B H^2 \sigma_y && \text{maximum moment for linear response} \\ M_{\mathrm{max}} &= B H^2 \sigma_y (\text{all interface is yielded}) && \text{maximum moment of the interface} \end{aligned}$$

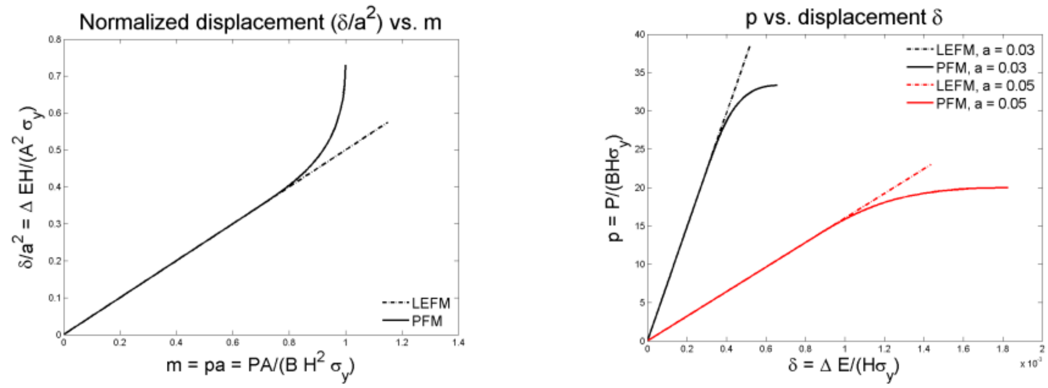
To determine the deflection Δ at the tip of the crack we employ relations between $M(x)$ and $\frac{\mathrm{d}^2\Delta(x)}{\mathrm{d}X^2}$ as follows:

$$\frac{\mathrm{d}^2\Delta(x)}{\mathrm{d}X^2} = \begin{cases} \dfrac{M(x)}{EI} & M(x) < M_{lmax} \\ \dfrac{\sigma_y}{HE} \dfrac{1}{\sqrt{3}\sqrt{1 - M(x)/M_{\mathrm{max}}}} & M_{lmax} < M(x) < M_{\mathrm{max}} \end{cases}$$

Note that $\Delta(x)$, $M(x)$ denote displacement and moment values along the beam while undecorated Δ and M denote their maximum values at the two end points of the crack. By locating the initiation position of plastic deformation in the beam and integrating (5) we obtain,

$$\delta = \frac{\Delta}{H}\frac{E}{\sigma_y} = a^2 f(m), \quad f(m) = \begin{cases} \dfrac{1}{2}m & m < \dfrac{2}{3} \\ \dfrac{20}{27m^2} - \dfrac{2}{3\sqrt{3}m^2}\sqrt{1-m}(2+m) & \dfrac{2}{3} < m < 1 \end{cases}$$

Equation (6) implies that when the applied moment $m = pa$ is small ($< \frac{2}{3}$ corresponding to M_{lmax}) the linear response holds between load and displacement. However, as m increases either through increasing load P or crack length A, the $P - \Delta$ relation is no longer linear.



(a) Relation between normalized displacement and moment,equation (6).

(b) $P - \Delta$ plots for sample crack lengths $A = aH$ based on (6).

Figure 3: Linear and nonlinear P − Δ relations for the crack problem in figure 2. The dash line LEFM curves show that for small "loading" (m, P), the actual P − Δ relation is linear

(a) Energy release rate $\mathbf{J}$: To characterize plastic fracture response of this crack, we need to evaluate energy release rate $J = G$. Since equation (6) is the $P - \Delta$ relation (in normalized form), we should be able to evaluate internal (strain) energy $U(\Delta, A) = \int_0^{\Delta} P(\bar{\Delta})\mathrm{d}\bar{\Delta}\Big|_{\text{fixed A}}$ or complimentary internal energy $U^*(P, A) = \int_0^{P} \Delta(\bar{P})\mathrm{d}\bar{P}\Big|_{\text{fixed A}}$ (note that the dummy parameters denoted by $\bar{(.)}$ are integrand integration variables). Subsequently, using one of the following equations $J = G = -\dfrac{1}{B}\dfrac{\mathrm{d}U(\Delta, A)}{\mathrm{d}A}\Big|_{fixed\,\Delta}$ or $J = G = \dfrac{1}{B}\dfrac{\mathrm{d}U^*(P, A)}{\mathrm{d}A}\Big|_{fixed\,P}$ we can evaluate J. Note that J is taken as the energy release rate per unit area of crack advance $\mathscr{A} = AB$ rather than crack length A.

Choosing the appropriate form of J in terms of U or U^* for this problem show that,

$$J(m) = \frac{\sigma_y^2 H}{E}\left\{\int_0^m f(\bar{m})\mathrm{d}\bar{m} + mf(m)\right\}$$

$$\frac{\Delta}{H}\frac{E}{\sigma_y} = a^2 f(m) \therefore \Delta = \frac{\sigma_y A^2}{EH} f(m)\ (1)$$

$$m = pa = \frac{PA}{\sigma_y BH^2} \therefore P = \frac{m\sigma_y BH^2}{A}$$

$$U^*(P, A) = \int_0^P \Delta(\bar{P})\mathrm{d}\bar{P}\Big|_{\text{fixed A}}\quad (2)\quad \text{and}\quad J = G = \frac{1}{B}\frac{\mathrm{d}U^*(P, A)}{\mathrm{d}A}\Big|_{fixed\,P}$$

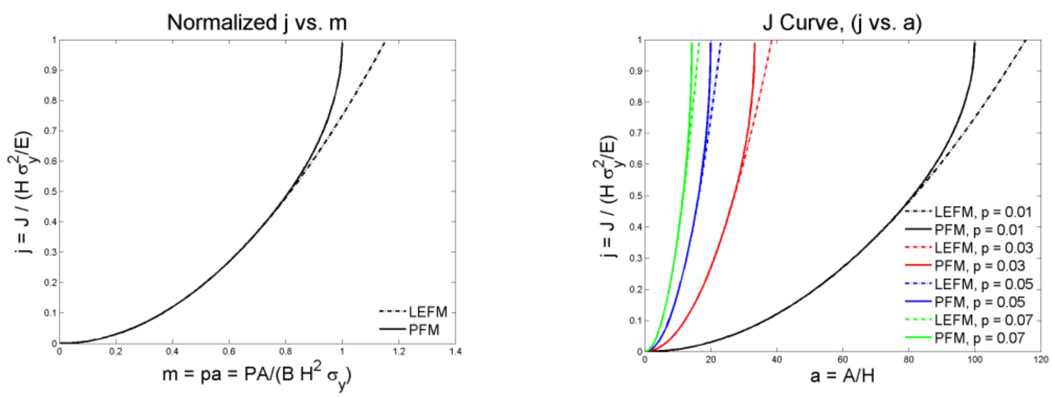
$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{\sigma_y BH^2}{A} \text{ and using (1), (2) becomes } U^*(m, A) = \int_0^m \frac{\sigma_y A^2}{EH} f(m)\frac{\sigma_y BH^2}{A}\mathrm{d}m\Big|_{\text{fixed A}}$$

$$\frac{1}{B}\frac{\mathrm{d}U^*}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}A} = \frac{\sigma_y^2 H}{E}\left\{\int_0^m f(\bar{m})\mathrm{d}\bar{m} + mf(m)\right\}, \text{ since P and A are fixed}$$

(b) LEFM vs. PFM, Small Scale Yielding (SSY): After evaluating (7) we can show (no need to prove (7) yields (8)) that normalized energy release rate j is equal to,

$$j = \frac{J}{\frac{\sigma_y^2 H}{E}} = \begin{cases} \frac{3}{4}m^2 & m < \frac{2}{3} \\ 1 - \frac{2}{\sqrt{3}}\sqrt{1-m} & \frac{2}{3} < m < 1 \end{cases}$$

This $J - m$ relation and its realization as $\mathbf{J}$ curve for specific load samples p are shown in (4).



(a) Energy release rate as a function of normalized moment $m = pa$, (8)

(b) J curve ($J vs. A$) plots for sample applied loads $P = p\sigma_y HB$ based on (8).

Figure 4: Energy release rate $\mathbf{J}$ as a function of normalized moment $m = pm$ and its realization for specific load values p. The LEFM solution does not take material yielding into account

i. What is the limiting m value, m_{tran}, below which LEFM solution can be used? For the geometry shown in 2, what is the transition load $P_{tran}(A)$ for a given crack length A for which LEFM solution can be employed?

Limiting m value is $m_{trans} = \frac{2}{3}$ below which LEFM solution can be used. Corresponding transition load is

$$P_{trans} = \frac{2}{3}\frac{\sigma_y BH^2}{A}$$

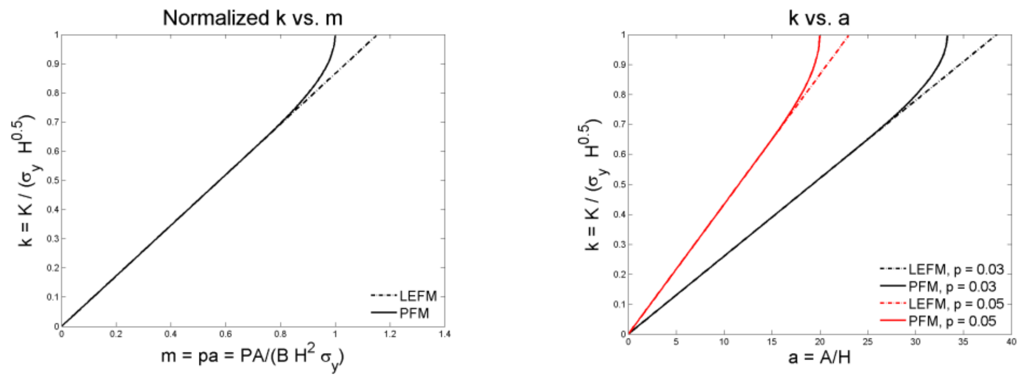
ii. Briefly (less than 2-4 sentences) explain why for $j > j_{tran}$ $(P > P_{tran}(A))$ plastic solution has a larger energy release rate?

Energy required to grow the crack is larger since the required energy to deform the region gets larger with the addition of plastic deformation.

iii. Since for LEFM $K = \sqrt{GE}$ (plane stress), the "effective" normalized K for this problem is,

$$k = \frac{K}{\sigma_y\sqrt{H}} = \begin{cases} \frac{\sqrt{3}}{2}m & m < \frac{2}{3} \\ \sqrt{1 - \frac{2}{\sqrt{3}}\sqrt{1-m}} & \frac{2}{3} < m < 1 \end{cases}$$

Consider a loading $P_1 = 0.05\sigma_y BH$, $A = 10H$. What is the stress intensity factor K_1 corresponding to this load? What is the stress intensity factor K_2 for $P_2 = 2P_1$ and same A? What is the relation between $2K_1$ and K_2 ? Using figure 5 (a) explain why the superposition principle (e.g., K of $2P$ is $2K$) does not hold here.



(a) Normalized stress intensity factor $k = \frac{K}{\sigma_y\sqrt{H}}$, (9)

(b) Stress intensity factor plots for sample applied loads $P = p\sigma_y HB$ based on (9).

Figure 5: "Effective" stress intensity factor computed from J.

Using $m = pa = \frac{PA}{\sigma_y BH^2}$

$$m = \frac{0.05\sigma_y BH10H}{\sigma_y BH^2}$$

$$m = 0.5$$

Since $m = 0.5 < 2/3$

```
K_1 = sqrt(3)/2*0.5
```

```
K_1 = 0.4330
```

$P_2 = 2P_1 \Rightarrow m_2 = 1$

Therefore $k_1 = 0.433$ **and** $k_2 = 1$**. The linear relationship does not hold for PFM since** $m_{trans} > 2/3$**, corresponding to the transition between plastic and fracture failure mechanisms.**

(c) Critical load P_{cr} and displacement Δ_{cr} correspond to load and displacement values that the crack can start propagating for a given fracture resistance J_c. To determine P_{cr}, as done before, in R plot (e.g., 4(b)) we find the smallest load (for load control) or displacement (displacement control) value whose J curve intersect R curve for the initial crack length A_0. If $R(A)$ is constant $R(A) = J_c$, for linear regime ($j_c = \frac{J_c}{\frac{\sigma_y^2 H}{E}} < \frac{1}{3}$, cf. (8), we obtain,

$$J = J_c \Rightarrow j = \frac{3}{4}m^2 = j_c\left(m < \frac{2}{3}\text{linear branch of } j\right) \quad \Rightarrow \quad m_{cr} = p_{cr}a_0 = \frac{2}{\sqrt{3}}\sqrt{j_c} \Rightarrow$$

$$P_{cr} = BH\sigma_y p_{cr} = BH\sigma_y \frac{m_{cr}}{a_0} = BH\sigma_y \frac{2}{\sqrt{3}}\sqrt{j_c}\frac{H}{A_0} = \frac{2}{\sqrt{3}}\sqrt{j_c}\frac{BH^2}{A_0}\sigma_y$$

Note that P_{cr} depends on the initial crack length A_0. Similarly, by plugging m_{cr} in (6) we obtain Δ_{cr}, the critical displacement for crack propagation initiation. Δ_{cr} is either directly applied in displacement control loading or is the displacement corresponding to P_{cr} for load control setting. These values are summaries as follows,

$$P_{cr} = \frac{2}{\sqrt{3}}\sqrt{j_c}\frac{BH^2}{A_0}\sigma_y$$

$$\Delta_{cr} = \sqrt{\frac{j_c}{3}\frac{\sigma_y}{E}\frac{A_0^2}{H}}$$

for $m < \frac{2}{3}\left(j_c < \frac{1}{3}\right)$.

i. Evaluate P_{cr}, Δ_{cr} for the nonlinear range $1 > m > \frac{2}{3}\left(1 > j_c > \frac{1}{3}\right)$ in terms of j_c and m_{cr}.

Using the relationship between j and m:

$$j = 1 - \frac{2}{\sqrt{3}}\sqrt{1-m}$$

$$\frac{\sqrt{3}}{2}(1-j) = \sqrt{1-m}$$

$$\frac{3}{4}(1-j)^2 = 1-m$$

$$m = 1 - \frac{3}{4}(1-j)^2$$

$$P_{cr} = BH\sigma_y p_{cr} = BH\sigma_y \frac{m_{cr}}{a_0} = BH\sigma_y\left((1 - \frac{3}{4}(1-j_{cr})^2\right)\frac{H}{A_0} = \left(1 - \frac{3}{4}(1-j_{cr})^2\right)\frac{BH^2}{A_0}\sigma_y$$

Using (6)

$$\delta_{cr} = \frac{\Delta_{cr}}{H}\frac{E}{\sigma_y} = a_0^2\left(\frac{20}{27m_{cr}^2} - \frac{2}{3\sqrt{3}\,m_{cr}^2}\sqrt{1-m_{cr}}\,(2+m_{cr})\right), \quad a_0 = \frac{A_0}{H}$$

$$\delta_{cr} = \frac{\Delta_{cr}}{H}\frac{E}{\sigma_y} = \frac{A_0^2}{H^2}\left(\frac{20}{27m_{cr}^2} - \frac{2}{3\sqrt{3}\,m_{cr}^2}\sqrt{1-m_{cr}}\,(2+m_{cr})\right)$$

$$\Delta_{cr} = \frac{\sigma_y A_0^2}{EH}\left(\frac{20}{27m_{cr}^2} - \frac{2}{3\sqrt{3}\,m_{cr}^2}\sqrt{1-m_{cr}}\,(2+m_{cr})\right)$$

$$\Delta_{cr} = \frac{\sigma_y A_0^2}{EH}\left(\frac{20}{27\left(1-\frac{3}{4}(1-j)^2\right)^2} - \frac{2}{3\sqrt{3}\left(1-\frac{3}{4}(1-j)^2\right)^2}\sqrt{1-\left(1-\frac{3}{4}(1-j)^2\right)}\left(2+\left(1-\frac{3}{4}(1-j)^2\right)\right)\right)$$

$$\Delta_{cr} = \frac{\sigma_y A_0^2}{EH}\left(\frac{20}{27\left(1-\frac{3}{4}(1-j)^2\right)^2} - \frac{2}{3\sqrt{3}\left(1-\frac{3}{4}(1-j)^2\right)^2}\sqrt{\frac{3}{4}(1-j)^2}\left(3-\frac{3}{4}(1-j)^2\right)\right)$$

$$\Delta_{cr} = \frac{\sigma_y A_0^2}{EH}\left(\frac{20}{27\left(1-\frac{3}{4}(1-j)^2\right)^2} - \frac{1-j-\frac{1}{4}(1-j)^3}{\left(1-\frac{3}{4}(1-j)^2\right)^2}\right)$$

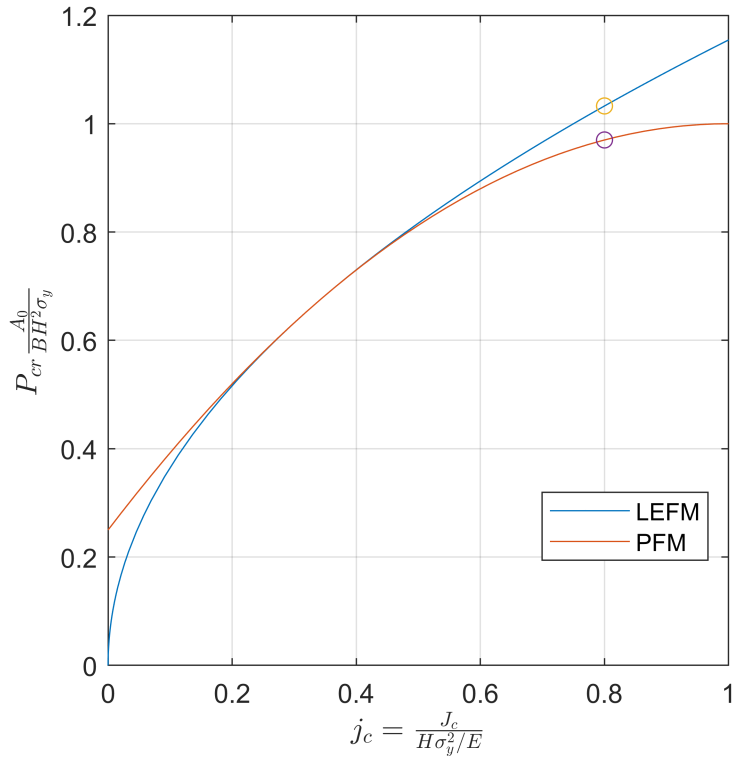
ii. Combining the solution from (11a) and your solution for $1 > m > \frac{2}{3}$, plot P_{cr} in the form $p_{cr}a_0 = P\frac{A_0}{BH^2\sigma_y}$ versus $j_c = \frac{J_c}{H\sigma_y^2/E}$ for the entire range $j_c = 0$ to 1 . In addition to P_{cr} from PFM, add the P_{cr} that you would have obtained from LEFM analysis for the entire $j_c \in [0 \quad 1]$ using (11a).

```
P_cr_lfm = @(j) 2*sqrt(j/3);
P_cr_pfm = @(j) 1-3/4*(1-j)^2;

figure('Position',[100 100 400 400])
hold on
fplot(P_cr_lfm,[0 1])
fplot(P_cr_pfm,[0 1])
```

```
Warning: Function behaves unexpectedly on array inputs. To improve performance, properly vectorize your
function to return an output with the same size and shape as the input arguments.
```

```
scatter(0.8,P_cr_lfm(0.8))
scatter(0.8,P_cr_pfm(0.8))
box on; grid on; legend('LEFM','PFM','Location','best')
ylabel('$ P_{cr} \frac{A_0}{B H^2 \sigma_y}$','Interpreter','latex')
xlabel('$j_c=\frac{J_c}{H \sigma_y^2 / E}$','Interpreter','latex')
hold off
```



iii. For what ranges of j_c, P_{cr} from LEFM and PFM analysis are different and in that range is PFM P_{cr} smaller or larger than that of LEFM analysis. Explain (less than 2-3 sentences) why of PFM is smaller or larger than that of LEFM.

```
syms j
j_cr = 2*sqrt(j/3) == 1-3/4*(1-j)^2
```

j_cr =

$$2\sqrt{\frac{j}{3}} = 1 - \frac{3\,(j-1)^2}{4}$$

```
solve(j_cr)
```

ans =

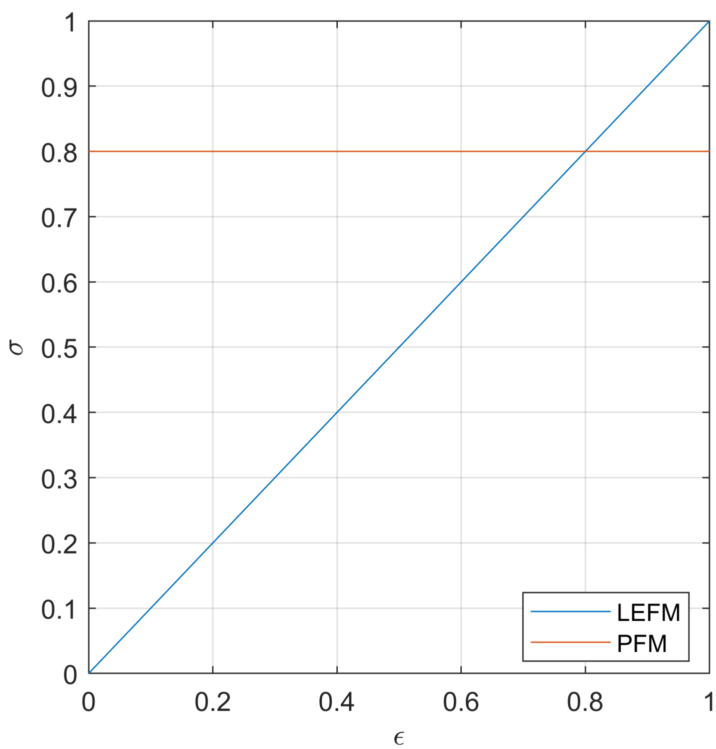
$$\frac{1}{3}$$

```
clear j
```

P_{cr} for $j < 1/3$ is larger for LEFM and P_{cr} for $1/3 < j < 1$ is larger for PFM. Since $J = G$, large j_c corresponds to high fracture toughness, thus the response will be dominated by plastic mechanism. We can see that the normalized P_c is approacing to 1. However, for lower j_c values, material response is mostly elastic, thus, LEFM aproach is more suitable. For zero load case, the curve for PFM is not going to zero since the material is assumed perfectly plastic.

Example stress-strain curves for perfectly elastic and perfectly plastic materials are given below.

```
x_min = 0;
x_max = 1;
figure('Position',[100 100 400 400])
hold on
plot(x_min:0.1:x_max,(0:0.1:x_max))
plot(x_min:0.1:x_max,(x_min:0.1:x_max)*0+0.8)
box on; grid on; legend('LEFM','PFM','Location','best')
ylabel('$\sigma$','Interpreter','latex')
xlabel('$\epsilon$','Interpreter','latex')
hold off
```



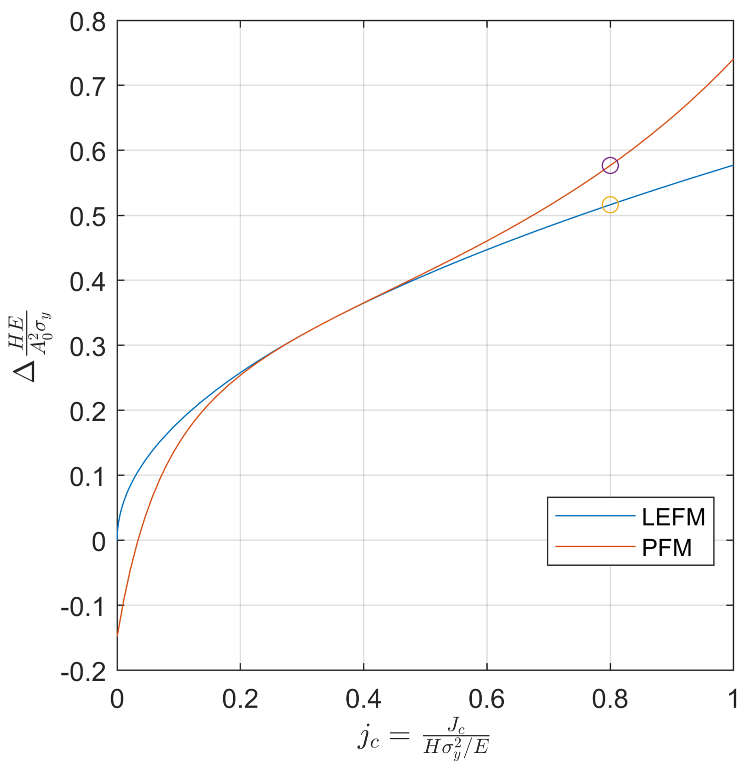
iv. Similarly, plot Δ_{cr} in the form $\frac{\delta_{cr}}{a_0} = \Delta \frac{HE}{A_0^2 \sigma_y}$ versus $j_c = \frac{J_c}{H\sigma_y^2/E}$ for $j_c \in [0 \quad 1]$ for both PFM and LEFM solutions using (11b) and your solution.

```
P_cr_lfm = @(j) sqrt(j/3);
P_cr_pfm = @(j) 20/(27*(1-3/4*(1-j)^2)^2)-(1-j-1/4*(1-j)^3)/(1-3/4*(1-j)^2)^2;

figure('Position',[100 100 400 400])
hold on
fplot(P_cr_lfm,[0 1])
fplot(P_cr_pfm,[0 1])
```

```
Warning: Function behaves unexpectedly on array inputs. To improve performance, properly vectorize your
function to return an output with the same size and shape as the input arguments.
```

```
scatter(0.8,P_cr_lfm(0.8))
scatter(0.8,P_cr_pfm(0.8))
box on; grid on; legend('LEFM','PFM','Location','best')
ylabel('$\Delta \frac{H E}{A_0^2 \sigma_y}$','Interpreter','latex')
xlabel('$j_c=\frac{J_c}{H \sigma_y^2 / E}$','Interpreter','latex')
hold off
```



And the critical point:

```
syms j
j_cr = sqrt(j/3) == 20/(27*(1-3/4*(1-j)^2)^2)-(1-j-1/4*(1-j)^3)/(1-3/4*(1-j)^2)^2
```

```
j_cr =
```

$$\sqrt{\frac{j}{3}} = \frac{20}{27\left(\frac{3\ (j-1)^2}{4} - 1\right)^2} - \frac{\frac{(j-1)^3}{4} - j + 1}{\left(\frac{3\ (j-1)^2}{4} - 1\right)^2}$$

```
solve(j_cr)
```

```
ans =
```

$$\frac{1}{3}$$

```
clear j
```

Similar to the load contol case, curves are intersecting at $j_{cr} = 1/3$

v. Compare Δ_{cr} from LEFM and PFM and comment on in which range they are different and briefly explain the cause of difference. You can refer to figure 6 for the explanation of your results.

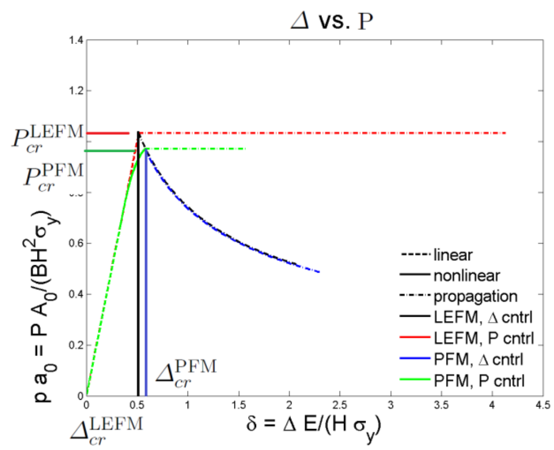


Figure 6: P_{cr} and Δ_{cr} from LEFM and PFM analysis of the crack with initial length A_0 for $j_c = 0.8$.

The differences between LEFM and PFM are distinct around critical values. LEFM can't represent the LSY for high fracture toughness, thus providing higher critical values. For P control, LEFM gives a higher critical load, while for control, PFM gives a higher value, as shown above. Discrepancy will increase as j_0 increases.